On Inequalities Obtained by Numerical Methods in a Terminal Control Problem Described by an Integral Equation

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***Abstract*: The issue of terminal control problem is very important in transport logistics. It provides more efficient control of cargo and passenger flows, and also helps optimize costs. Terminal control problem is important for choosing optimal cargo transportation routes, timely and safe delivery of goods. This regulates transport flows, reduces costs and makes the overall logistics process more efficient. Of course, the issue of terminal control problem is the process of optimal planning and control of cargo or passenger flows. The main goal here is to maximize the efficient use of resources and minimize costs. This also helps save time and resources. In this work, new integral inequalities for the trajectory in the case of terminal control problem described by an integral equation are obtained using numerical methods, which is one of the new mathematical effects obtained in the mathematical theory of optimal control.**

***Keywords*: optimal control, terminal control, maximum principle, integral equation, integral inequalities.**

I. INTRODUCTION

Different classes of lumped and distributed parameter optimal control problems described by ordinary and partial differential equations are studied in detail from various aspects [1-9].The paper shows that the terminal control problem described by an integral equation from a special class of optimal control problems with lumped parameters can be described by an integral inequality under certain assumptions.

Suppose a controlled object

$$\dot{x}(t) = \Phi(t, x(t), u(t)), t \in [t_0, t_1], \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

it is illustrated by the Cauchy problem. According to the solution $x(t)$ of equation (1) satisfying the initial condition (2) it is required to find a controller $u(t) \in U_m$ such that

$$T(u) = \varphi(x(t_1)) \to \min, \tag{3}$$

Usually, a set of piecewise continuous or measurable bounded functions is taken as a class of possible controls U_m .These are the functions $\Phi(t, x(t), u(t)), \varphi(x)$ given above. Problem (1)-(3) is called the terminal control problem.

Suppose that controller $u(t) \in U_m$ and corresponding to it function $x(t)$, $t \in [t_0, t_1]$ is the solution of equation (1) satisfying the initial condition (2). Then the pair $(u(t), x(t))$ is called a process. According to this process of equation

$$\dot{\Psi}(t) = -H_x(t, x(t), \Psi(t), u(t)), \tag{4}$$

let's look at the solution that satisfies the condition

$$\Psi(t_1) = -\varphi_x(x(t_1)), \tag{5}$$

here

$$H(t, x(t), \Psi(t), u(t)) = \Psi(t)\Phi(t, x(t), u(t))$$

it is called the Hamiltonian function. Problem (4)-(5) is called a supplementary problem to terminal control problem (1)-(3).
Definition. Certain controller $u^*(t) \in U_m$ and for an arbitrary controller $u(t) \in U_m$ if $T(u^*) \le T(u)$, then controller $u^*(t)$ is called optimal controller due to its functionality $T(u)$. In this case, the pair $(u^*(t), x(t))$ is called an optimal process.

As is known from the mathematical theory of optimal control, that the maximum principle for the terminal control problem (1)-(3) is as follows:
Theorem (Maximum principle) Assume that the $u(t)$ optimal controller, functions $x(t)$ and $\Psi(t)$ respectively, are solutions of problems (1)-(2) and (4)-(5) corresponding to this controller. Then for all points $t \in [t_0, t_1]$ the maximum condition is paid:

$$H(t, x(t), \Psi(t), u(t)) = \max_{\nu \in U_m} H(t, x(t), \Psi(t), \nu), \tag{6}$$

Note that the maximum condition in (6) can be written in the following form

$$H(t, x(t), \Psi(t), \nu) - H(t, x(t), \Psi(t), u(t)) \le 0, \quad t \in [t_0, t_1], \nu \in U_m, \tag{7}$$

Here inequality (7) shows the maximum condition of the terminal control problem.

The above are known facts related to the terminal control problem (1)-(3).In this work, we study the obtaining of new mathematical effects in the time slice $t \in [t_0, t_1]$ for the terminal control problem (1)-(3).

II. PROBLEM STATEMENT

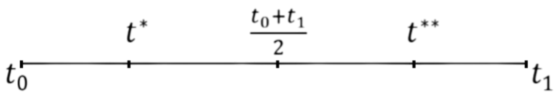
The goal is to obtain new integral inequalities for the trajectory $x(t),\ t \in [t_0, t_1]$ in the terminal control problem (1)-(3). To do this, we will use the following auxiliary fact.

Auxiliary fact: Study of the problem of terminal control with division of time interval by numerical method.

The Fibonacci sequence is a sequence in which each number is the sum of the two previous numbers. Fibonacci numbers are defined by their equalities: $F_1 = F_2 = 1, F_{n+2} = F_n + F_{n+1}, n \in N.$ Let's show some of these number sequences: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89… We will use the Fibonacci method, which is one of the numerical methods for dividing a time interval. Now, using the Fibonacci sequence of numbers, we divide the time interval $[t_0, t_1]$according to the following rule:

$$t^* = t_0 + \frac{(t_1 - t_0)F_n}{F_{n+2}}, n \in N, n \geq 2$$

$$t^{**} = t_0 + \frac{(t_1 - t_0)F_{n+1}}{F_{n+2}}, n \in N, n \geq 2$$



These points are symmetrical about the center of the time interval $[t_0, t_1]$, point t^* to the left of the middle of the time interval, the point t^{**} is on the right:

$$t_0 < t^* < \frac{t_0 + t_1}{2} < t^{**} < t_1, \tag{8}$$

Note that, knowing one of the points t^* and t^{**}, the other can be found using the following formula:

$$t^* + t^{**} = t_0 + t_1.$$

Really,

$$t^* + t^{**} = t_0 + \frac{(t_1 - t_0)F_n}{F_{n+2}} + t_0 + \frac{(t_1 - t_0)F_{n+1}}{F_{n+2}} =$$

$$= 2t_0 + \frac{(t_1 - t_0)(F_n + F_{n+1})}{F_{n+2}} = 2t_0 + \frac{(t_1 - t_0)F_{n+2}}{F_{n+2}} =$$

$$= 2t_0 + t_1 - t_0 = t_0 + t_1.$$

III. THE TERMINAL CONTROL PROBLEM DESCRIBED BY AN INTEGRAL EQUATION

According to the Newton-Leibniz formula, known from integral calculus, the trajectory $x(t),\ t \in [t_0, t_1]$ can be described as follows:

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)d\tau, \tag{9}$$

In fact, formula (9) shows that the trajectory is an absolutely continuous function. An absolutely continuous function, as is known, is a continuous function.

If we consider the right side of equation (1) and the right side of the initial condition (2) in formula (9), we obtain:

$$x(t) = x_0 + \int_{t_0}^{t} \Phi(\tau, x(\tau), u(\tau))d\tau, \tag{10}$$

Thus, equation (1), which describes the controlled process, was reduced to the Volterra integral equation of the second kind in the form (10) with initial conditions (2). Since this integral equation contains both the differential equation (1) and the initial condition (2), then for each fixed control function $u(t) \in U_m$ (10) we will call the integral equation equivalent to the Cauchy problem (1)-(2). In other words, equation (10) is called an equivalent integral equation.

It is clear that the upper limit of the variable integral in the integral equation (10) can be written as follows:

$$\int_{t_0}^{t} \Phi(\tau, x(\tau), u(\tau))d\tau = \int_{t_0}^{t_1} \theta(t - \tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \tag{11}$$

Here,

$$\theta(t) = \begin{cases} 1, t > 0 \\ 0, t \leq 0 \end{cases}$$ -this is the Heaviside function.

Thus, using equality (11), the integral equation (10) can be written in the form of the following equivalent integral equation:

$$x(t) = x_0 + \int_{t_0}^{t_1} \theta(t - \tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \tag{12}$$

Thus, the terminal problem, described by the integral equation, can be expressed as follows:

It is required to find the control function $u(t) \in U_m$ corresponding to the solution $x(t)$ of the integral equation (12), which gives a minimum to the functional (3). Thus, instead of the terminal control problem (1)-(3), we can speak of an equivalent terminal control problem in integral form (3), (12).

IV. New Integral Inequalities for the Trajectory in the Studying the Terminal Control Problem

Theorem1. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) < x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(\frac{t_0+t_1}{2}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(t_1) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (13)$$

Proof. If using inequality (8), we assume that the function is strictly monotonically increasing, then we can write:

$$x(t_0) < x(t^*) < x(\frac{t_0+t_1}{2}) < x(t^{**}) < x(t_1), \quad (14)$$

Then, taking into account inequality (14) in the integral equation (12), we obtain:

$$x(t) = x_0 + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau =$$
$$= x(t_0) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(\frac{t_0+t_1}{2}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau <$$
$$< x(t_1) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (15)$$

As can be seen, the integral inequalities obtained in the formula (15) show the truth of the formula (13) in Theorem 1.

Theorem2. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) > x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau >$$
$$> x(\frac{t_0+t_1}{2}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau >$$
$$> x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau >$$
$$> x(t_1) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 2 is carried out similarly to Theorem 1.

Theorem3. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) < \frac{x(t^*) + x(\frac{t_0+t_1}{2})}{2} +$$
$$+ \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (16)$$

Proof. According to formula (13), the following is true:

$$x(t) < x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (17)$$

$$x(t) < x(\frac{t_0+t_1}{2}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (18)$$

If we add inequalities (17) and (18) side by side and divide both parts by 2, we get formula (16).

Theorem4. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) > \frac{x(t^*) + x(\frac{t_0+t_1}{2})}{2} +$$
$$+ \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 4 is carried out similarly to Theorem 3.

Theorem5. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) < \frac{x(t^*) + x(t^{**})}{2} + \\ + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (19)$$

Proof. According to formula (13), the following is true:

$$x(t) < x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (20)$$

$$x(t) < x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (21)$$

If we add inequalities (20) and (21) side by side and divide both parts by 2, we get formula (19).

Theorem6. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) > \frac{x(t^*) + x(t^{**})}{2} + \\ + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 6 is carried out similarly to Theorem 5. Note that it is possible to prove other theorems similar to Theorem 3, Theorem 4, Theorem 5 and Theorem 6.

Theorem7. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) < \frac{x(t^*) + x(\frac{t_0 + t_1}{2}) + x(t^{**})}{3} + \\ + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (22)$$

Proof. According to formula (13), the following is true:

$$x(t) < x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (23)$$

$$x(t) < x(\frac{t_0 + t_1}{2}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (24)$$

$$x(t) < x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (25)$$

If we add inequalities (23), (24) and (25) side by side and divide both parts by 3, we get formula (22).

The theorem has been proven.

Theorem8. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) > \frac{x(t^*) + x(\frac{t_0 + t_1}{2}) + x(t^{**})}{3} + \\ + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 8 is carried out similarly to Theorem 7.

Theorem9. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) < \frac{x(t^*) + x(t^{**}) + x(t_1)}{3} + \\ + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (26)$$

Proof. According to formula (13), the following is true:

$$x(t) < x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (27)$$

$$x(t) < x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (28)$$

$$x(t) < x(t_1) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (29)$$

If we add inequalities (27), (28) and (29) side by side and divide both parts by 3, we get formula (26).

Theorem10. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) > \frac{x(t^*) + x(t^{**}) + x(t_1)}{3} +$$
$$+ \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 10 is carried out similarly to Theorem 9.The proof of Theorem 6 is carried out similarly to Theorem 5. Note that it is possible to prove other theorems similar to Theorem 7, Theorem 8, Theorem 9 and Theorem 10.

Theorem11. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) < \frac{x(t^*) + (\frac{t_0 + t_1}{2}) + x(t^{**}) + x(t_1)}{4} + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (30)$$

Proof. According to formula (13), the following is true:

$$x(t) < x(t^*) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (31)$$

$$x(t) < x(\frac{t_0 + t_1}{2}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (32)$$

$$x(t) < x(t^{**}) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (33)$$

$$x(t) < x(t_1) + \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau, \quad (34)$$

If we add inequalities (31), (32), (33) and (34) side by side and divide both parts by 4, we get formula (30).

Theorem12. If the trajectory $x(t), t \in [t_0, t_1]$ in the terminal control problem in integral form (3), (12) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) > \frac{x(t^*) + (\frac{t_0 + t_1}{2}) + x(t^{**}) + x(t_1)}{4} +$$
$$+ \int_{t_0}^{t_1} \theta(t-\tau)\Phi(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 12 is carried out similarly to Theorem 11.

CONCLUSION

Using the example of a terminal control problem described by an integral equation, it is shown that this problem can be described with integral inequalities under the assumption of strict monotonicity of the trajectory.

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