

Constructing an Approximate Solution to the Problem of Vibrations of Viscoelastic Systems

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Abstract—The paper deals with a boundary value problem of vibrations of a viscoelastic system that are described by Volterra type partial integro-differential equations. Here we propose an approach using the Laplace integral transform, which allows to construct a solution to the integro-differential equation in the form of a series, the first term of which is the solution to this equation obtained by the averaging method. For a specific kernel, the effectiveness of this approach is shown, and the influence of various parameters of the equations and initial conditions on the error of the obtained solution is evaluated.

Keywords—viscoelasticity, kernel, rheology image, original, Volterra equation, integral transform

I. INTRODUCTION

The importance of studying the theory of vibrations of viscoelastic systems recent years has been outlined in many works [1,3].

Without correct knowledge of the theory of vibrations of these systems with rheological properties, the successful production of such structures will be practically impossible and unsafe. The study of such problems mathematically reduces to solving Volterra type partial integro-differential equations with appropriate boundary and initial conditions.

Unfortunately, the construction of solutions to problems of this class is associated with a number of difficulties due to nonlinear expressions and integrals of unknown relaxation functions. The proposed approach allows to construct the solution to the integro-differential equation of vibrations of viscoelastic systems in the form of a series the first term of which is a solution obtained by the averaging method [2,4].

For a specific kernel the effectiveness of this approach is shown, the influence of various parameters of equations and initial conditions on the error of the obtained solution is evaluated.

II. PROBLEM STATEMENT

Let us consider forced lateral vibrations of a viscoelastic beam whose end are supported by the described equation:

$$\frac{M}{B} \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^4 w(x,t)}{\partial x^4} = \int_0^t \Gamma(t-\tau) \frac{\partial^4 w(x,t)}{\partial x^4} d\tau + \frac{Q}{B}, \quad (1)$$

where $w(x,t)$ is displacement, Q is a load per unit beam length, M is a mass per unit beam length, B is a reduced rigidity, $\Gamma(t)$ is a relaxation kernel and we assume that this kernel is proportional to some positive small parameters. In this case the following conditions are fulfilled [1]

$$0 \leq \int_0^t \Gamma(\tau) d\tau < 1, \quad \Gamma(t) \geq 0 \text{ for any time } t \quad (2)$$

We accept initial conditions in the form

$$w(x,0) = z_0, \quad \frac{\partial w(x,0)}{\partial t} = z'_0; \quad 0 \leq x \leq \ell \quad (3)$$

where ℓ is the length of the beam.

Condition (2) means that for $t < 0$ the system is at rest.

By the problem condition the boundary conditions will be

$$\begin{aligned} \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad \frac{\partial^3 w(x,t)}{\partial x^3} = 0 \quad \text{npu } x = 0 \\ \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad \frac{\partial^3 w(x,t)}{\partial x^3} = 0 \quad \text{npu } x = \ell. \end{aligned} \quad (4)$$

The problem is mathematically reduced to determining the solutions to the system of equations (1)-(3) in $D = [0, \ell] \times [0, T]$.

III. PROBLEM SOLUTION

Note that by one of the known methods, the dynamic problem of viscoelasticity can be reduced to the system of integro-differential equations in time [2,4]. By means of one of these methods, equation (1) can be led to the system of integro-differential equations in the form:

$$\frac{d^2 V_0}{dt^2} + \lambda_n^2 V_n = \varepsilon \lambda_n^2 \int_0^t \Gamma(t-\tau) V_n(\tau) d\tau + \varepsilon_1 F_n(t); \quad (n = 1, 2, \dots) \quad (5)$$

where $V_n(t)$ are amplitude functions of time, ε_1 and $F_n(t)$ are amplitudes of small external loads, $\lambda_n = \frac{\pi_n}{\ell^2} \sqrt{\frac{B}{M}}$.

Applying the Laplace integral transform to the equation (4), allowing for appropriate initial conditions and omitting indices for simplicity of records, we obtain:

$$\bar{V}(p) = \frac{pu_0 + V_0}{p^2 + \lambda^2 - \varepsilon\lambda^2\bar{\Gamma}(p)} + \frac{\bar{F}(p)}{p^2 - \lambda^2 - \varepsilon\lambda^2\bar{\Gamma}(p)} \quad (6)$$

where p is a Laplace transform operator, $\bar{V}(p)$ and $\bar{\Gamma}(p)$ are the images of the functions $V(t)$ and $\Gamma(t)$, respectively.

To calculate the original of solutions, we represent equation (6) in the form of a series:

$$\bar{V}(p) = \frac{pu_0 + V_0}{p^2 + \lambda^2} \sum_{n=0}^{\infty} \left(\frac{\varepsilon\lambda^2\bar{\Gamma}(p)}{p^2 + \lambda^2} \right)^n + \frac{\bar{F}(p)}{p^2 + \lambda^2} \sum_{n=0}^{\infty} \left(\frac{\varepsilon\lambda^2\bar{\Gamma}(p)}{p^2 + \lambda^2} \right)^n \quad (7)$$

Transform the fraction $\frac{\varepsilon\lambda^2\bar{\Gamma}(p)}{p^2 + \lambda^2}$ as follows:

$$\frac{\varepsilon\lambda^2\bar{\Gamma}(p)}{p^2 + \lambda^2} = \frac{\varepsilon\lambda^2\Gamma_c - \varepsilon\lambda p\Gamma_s - \varepsilon\lambda^2(p^2 + \lambda^2)\bar{A}(p)}{p^2 + \lambda^2} \quad (8)$$

where $A(t) = \int_t^\infty \Gamma(\tau) \sin(t - \tau) d\tau$; $\Gamma_s = \int_0^\infty \Gamma(\tau) \sin \lambda \tau d\tau$;

$$\Gamma_c = \int_0^\infty \Gamma(\tau) \cos \lambda \tau d\tau.$$

Due to conditions (2) the value $|\varepsilon\lambda^2\bar{A}(p)|$ will be small if the time t is large enough. Therefore, in the formula (8) we can ignore the term $|\varepsilon\lambda^2\bar{A}(p)|$ valid for rather large values of time. Allowing for this condition, substituting (8) in (7) and summing over the series, we find:

$$\bar{V}(p) = \frac{pu_0 + V_0}{\bar{a}p - \varepsilon\lambda^2 d} + \frac{\bar{F}(p)}{\bar{a}p - \varepsilon\lambda^2 d}, \quad (9)$$

where $\bar{a}(p) = (p + \frac{1}{2}\varepsilon\lambda\Gamma_s)^2 + \lambda^2(1 - \frac{1}{2}\varepsilon\Gamma_c)^2$

$$d = \frac{1}{4}(\Gamma_c^2 + \Gamma_s^2)$$

Assuming, $\left| \frac{d}{\bar{a}(p)} \right| < 1$, we represent the solution (9)

in the form:

$$\bar{V}(p) = \frac{pu_0 + V_0}{\bar{a}(p)} [1 + \varepsilon\lambda^2 \frac{d}{\bar{a}(p)} + \varepsilon^2\lambda^4 \frac{d^2}{\bar{a}^2(p)} + \dots] + \frac{\bar{F}(p)}{\bar{a}(p)} [1 + \varepsilon\lambda^2 \frac{d}{\bar{a}(p)} + \varepsilon\lambda^4 \frac{d^2}{\bar{a}^2(p)} + \dots] \quad (10)$$

Here the first sum corresponds to free vibration, the second sum to the forced vibration of the system, that are denoted by $\bar{g}(p)$ and $\bar{\psi}(p)$ respectively, i.e.

$$\bar{g}(p) = \frac{pu_0 + V_0}{\bar{a}(p)} [1 + \varepsilon\lambda^2 \frac{d}{\bar{a}(p)} + \varepsilon^2\lambda^4 \frac{d^2}{\bar{a}^2(p)} + \dots] \quad (11)$$

$$\bar{\psi}(p) = \frac{\bar{F}(p)}{\bar{a}(p)} [1 + \varepsilon\lambda^2 \frac{d}{\bar{a}(p)} + \varepsilon\lambda^4 \frac{d^2}{\bar{a}^2(p)} + \dots] \quad (12)$$

Calculating the originals of each approximation, we find

$$V(t) = \sum_{i=1}^{\infty} [g_i(t) + \psi_i(t)]$$

where the functions $g_i(t)$ and $\psi_i(t)$ are determined as follows:

$$g_1(t) = \exp(-\frac{1}{2}\varepsilon\Gamma_s\lambda t) [u_0 \cos \lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)t +$$

$$+ \frac{V_0 - \frac{1}{2}\varepsilon\lambda\Gamma_s}{\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)} \sin \lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)t]$$

$$g_2(t) = \varepsilon\lambda^2 \int_0^t g_1(t - \tau) \chi(\tau) d\tau, \dots, g_n(t) = \varepsilon\lambda^2 \int_0^t g_{n-1}(t - \tau) \chi(\tau) d\tau$$

$$\psi_1(t) = \frac{1}{\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)} \int_0^t \Gamma(t - \tau) \ell^{\frac{1}{2}\varepsilon\lambda\Gamma_s\tau} \sin \lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)\tau d\tau$$

$$\psi_2(t) = \varepsilon\lambda^2 \int_0^t \psi_1(t - \tau) \chi(\tau) d\tau, \dots, \chi_n(t) = \varepsilon\lambda^2 \int_0^t \chi_{n-1}(t - \tau) \chi(\tau) d\tau$$

where

$$\chi(t) = L^{-1} \left\{ \frac{d}{\bar{a}(p)} \right\} = \frac{d}{\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)} \ell^{-\frac{1}{2}\varepsilon\lambda\Gamma_s t} \sin \lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)t;$$

Then, for the first approximation of solutions to the stated problem we determine

$$V_1(t) = D \exp(-\frac{1}{2}\varepsilon\lambda\Gamma_s t) \sin[\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)t + \theta] + N \int_0^t F(t - \tau) \exp(-\frac{1}{2}\varepsilon\lambda\Gamma_s t) \sin(1 - \frac{1}{2}\varepsilon\Gamma_c)\tau d\tau \quad (13)$$

$$\text{where } D = \sqrt{\left(\frac{V_0^2 - \frac{1}{2}\varepsilon\lambda\Gamma_s}{\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)} \right)^2 + u_0^2}; \quad \theta = \arctg \frac{u_0\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)}{V_0 - \frac{1}{2}\varepsilon\lambda\Gamma_s}$$

$$N = \frac{1}{\lambda(1 - \frac{1}{2}\varepsilon\Gamma_c)}.$$

In a similar way, we can determine subsequent approximations of the solutions.

From equation (13) it is seen that the oscillation decays exponentially over time.

Conclusions.

1. Using the method of operational calculus, the solution to the integro-differential equation of vibration is constructed in the form of a series, the first term of which coincides with the solution of this equation obtained by the averaging method

2. It was obtained that the vibration of the system decays over time according to the exponential law with the coefficient $(-\frac{1}{2} \varepsilon \lambda \Gamma_s)$.

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