# Boundary Control Problem for the Equation of Vibration of a Thin Plate

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***Abstract*—In this paper, we consider a boundary-value problem with boundary control for the equation of vibrations of the thin-plate. It is known that many practical problems are described by the equation of oscillation of a thin plate, but the controllability problems with this equation have almost never been studied. One of these questions is investigated in the present work. After determination the controllability, by introducing an auxiliary boundary value problem and using the result of the Han-Banach theorem, the controllability of the system under consideration is proved.**

***Keywords*— *oscillation equation, thin plate, boundary control, weak solution, controllability, observability***

## I. Introduction

It is known that a number of processes in physics and technology are described by fourth-order partial differential equations. For example, the equations of vibration of a rod, tuning fork, elastic plate, thin plate, etc. are such equations [1], [2]. Therefore, studies of optimal control and controllability problems in processes described by such equations are important tasks [3]. When control functions are boundary functions, learning control problems becomes too difficult. But we note that the problem of boundary control from theoretical and practical points of view is more natural compared to problems with distributed controls.

Note that in the works [4]–[7] some related control problems were considered. Namely, in [4] the problem of precise controllability of a three-dimensional linear-elastic thin plate with control on the boundary and inside the plate is considered, in [5] the problem of controlling transverse vibrations of a thin plate is considered, control actions are applied to the boundary of the plate, which fills a certain limited area on plane, and in [6], [7] the problem of boundary optimal control for the linear equation of oscillations of a thin plate was studied, and the existence and uniqueness theorem of optimal control was proved.

It should be noted that recently the problem of controllability of plate oscillations has been intensively studied [8]-[13].

In the work considered, one controllability problem is studied for the vibration equation of a thin plate.

## II. Problem Statement

Let the controlled process be described by the equation of oscillations of a thin plate

$$\frac{\partial^2 u}{\partial t^2} + a^2 \Delta^2 u = 0 \text{ в } Q = \Omega \times (0,T),$$
$$\Omega = (0,l_1)\times(0,l_2) \qquad (1)$$

with initial

$$u(x_1,x_2,0)=\varphi_0(x_1,x_2), \frac{\partial u(x_1,x_2,0)}{\partial t} = \varphi_1(x_1,x_2),$$
$$(x_1,x_2)\in\Omega \qquad (2)$$

and boundary conditions

$$u(0,x_2,t)=\upsilon_1^0(x_2,t),\ u(l_1,x_2,t)=\upsilon_2^0(x_2,t),$$
$$\frac{\partial u(0,x_2,t)}{\partial x_1}=\upsilon_1^1(x_2,t),\ \frac{\partial u(l_1,x_2,t)}{\partial x_1}=\upsilon_2^1(x_2,t),$$
$$(x_2,t)\in(0,l_2)\times(0,T),$$
$$u(x_1,0,t)=\upsilon_3^0(x_1,t),\ u(x_1,l_2,t)=\upsilon_4^0(x_1,t), \qquad (3)$$
$$\frac{\partial u(x_1,0,t)}{\partial x_2}=\upsilon_3^1(x_1,t),\ \frac{\partial u(x_1,l_2,t)}{\partial x_2}=\upsilon_4^1(x_1,t),$$
$$(x_1,t)\in(0,l_1)\times(0,T),$$

where $a^2, l_1, l_2, T$ - are the given positive numbers, $\Delta$ -is Laplace operator with respect to $x_1, x_2$, $\varphi_0(x_1,x_2)\in H^2(\Omega)$, $\varphi_1(x_1,x_2)\in L_2(\Omega)$ - are the given functions, $\upsilon_1^0(x_2,t)$, $\upsilon_2^0(x_2,t)$, $\upsilon_3^0(x_1,t)$, $\upsilon_4^0(x_1,t)$, $\upsilon_1^1(x_2,t)$, $\upsilon_2^1(x_2,t)$, $\upsilon_3^1(x_1,t)$, $\upsilon_4^1(x_1,t)$ - are the control functions.

The class of admissible controls is taken to be the space

$$U=\{\upsilon_1^0(x_2,t),\upsilon_2^0(x_2,t),\upsilon_3^0(x_1,t),\upsilon_4^0(x_1,t),$$
$$\upsilon_1^1(x_2,t),\upsilon_2^1(x_2,t),\upsilon_3^1(x_1,t),\ \upsilon_4^1(x_1,t),$$
$$\upsilon_i^0,\upsilon_i^1\in L_2\big((0,l_2)\times(0,T)\big), i=1,2,$$
$$\upsilon_i^0,\upsilon_i^1\in L_2((0,l_1)\times(0,T)), i=3,4\}$$

A weak solution to this problem is determined using transposition [3]: there is a unique function $u(\upsilon)\in L_2(Q)$, for which

$$\int_Q u(\upsilon)\left(\frac{\partial^2\varphi}{\partial t^2}+a^2\Delta^2\varphi\right)dx_1dx_2dt =$$
$$= a^2\Bigg(\int_0^T\int_0^{l_2}\upsilon_1^0(x_2,t)\frac{\partial\Delta\varphi(0,x_2,t)}{\partial x_1}dx_2dt$$
$$-\int_0^T\int_0^{l_2}\upsilon_2^0(x_2,t)\frac{\partial\Delta\varphi(l_1,x_2,t)}{\partial x_1}dx_2dt+$$
$$+\int_0^T\int_0^{l_2}\upsilon_1^1(x_2,t)\Delta\varphi(0,x_2,t)dx_2dt$$
$$-\int_0^T\int_0^{l_2}\upsilon_2^1(x_2,t)\Delta\varphi(l_1,x_2,t)dx_2dt+$$
$$+\int_0^T\int_0^{l_1}\upsilon_3^0(x_1,t)\frac{\partial\Delta\varphi(x_1,0,t)}{\partial x_2}dx_1dt$$
$$-\int_0^T\int_0^{l_1}\upsilon_4^0(x_1,t)\frac{\partial\Delta\varphi(x_1,l_2,t)}{\partial x_2}dx_1dt+$$
$$+\int_0^T\int_0^{l_1}\upsilon_3^1(x_1,t)\varphi(x_1,0,t)dx_1dt-$$
$$\int_0^T\int_0^{l_2}\upsilon_4^1(x_1,t)\Delta\varphi(x_1,l_2,t)dx_1dt\Bigg), \forall\varphi\in\Phi,$$

where

$$\Phi=\Bigg\{\varphi|D_x^p\varphi\in L_2(Q), |p|\le 4, \frac{\partial\varphi}{\partial t}\in L_2(Q), \frac{\partial^2\varphi}{\partial t^2}L_2(Q),$$
$$\varphi(x_1,x_2,T)=0, \frac{\partial\varphi(x_1,x_2,T)}{\partial t}=0, \varphi(0,x_2,t)=0,$$

$$\varphi(l_1,x_2,t)=0, \varphi(x_1,0,t)=0, \varphi(x_1,l_2,t)=0,$$
$$\frac{\partial\varphi(0,x_2,t)}{\partial x_1}=0, \frac{\partial\varphi(l_1,x_2,t)}{\partial x_1}=0, \frac{\partial\varphi(x_1,0,t)}{\partial x_2}=0,$$
$$\left.\frac{\partial\varphi(x_1,l_2,t)}{\partial x_2}=0, (x_1,x_2)\in\Omega\right\}.$$

**Definition.** A system whose state is defined as a solution to problem (1)-(3) is called controllable if the observation
$$\left(u(x_1,x_2,T), \frac{\partial u(x_1,x_2,T)}{\partial t}\right)$$
sweeps the space $L_2(\Omega)\times H^{-2}(\Omega)$, when the control
$$\upsilon=(\upsilon_1^0(x_2,t), \upsilon_2^0(x_2,t), \upsilon_3^0(x_1,t), \upsilon_4^0(x_1,t), \upsilon_1^1(x_2,t),$$
$$\upsilon_2^1(x_2,t), \upsilon_3^1(x_1,t), \upsilon_4^1(x_1,t))$$
runs through the space $U$, where $H^{-2}(\Omega)$ is the conjugate space to the space $H_0^2(\Omega)$, and
$$H_0^2(\Omega)=\left\{\chi(x_1,x_2)|\chi\in H^2(\Omega), \chi|_\Gamma=0, \left.\frac{\partial\chi}{\partial\nu}\right|_\Gamma=0\right\},$$
$\Gamma$ - is the outer normal to the boundary of $\Gamma$.

## III. Main Results

In this problem, the following theorem is proved:

**Theorem.** Under the above conditions imposed on these problems (1)-(3), this system is controllable

Proof. Let the vector $(\psi_0(x_1,x_2), \psi_1(x_1,x_2))$ be orthogonal to the image $U$ under the mapping
$$\upsilon\to\left(u(x_1,x_2,T), \frac{\partial u(x_1,x_2,T)}{\partial t}\right)$$
i.e.
$$\int_\Omega u(x_1,x_2,T)\psi_0(x_1,x_2)dx_1dx_2+$$
$$\int_\Omega \frac{\partial u(x_1,x_2,T)}{\partial t}\psi_1(x_1,x_2)dx_1dx_2=0, \quad (4)$$
where $\psi_0(x_1,x_2)\in H_0^2(\Omega)$, $\psi_1(x_1,x_2)\in L_2(\Omega)$.

We want to find out whether it will follow from this that $\psi_0(x_1,x_2)\equiv 0$, $\psi_1(x_1,x_2)\equiv 0$ [14].

Let us assume that the function $\xi(x_1,x_2,t)$ is a solution to the following auxiliary problem:
$$\frac{\partial^2\xi}{\partial t^2}+a^2\Delta^2\xi=0, Q=\Omega\times(0,T),$$
$$\Omega=(0,l_1)\times(0,l_2), \quad (5)$$
$$\xi(x_1,x_2,T)=-\psi_1(x_1,x_2), \frac{\partial\xi(x_1,x_2,T)}{\partial t}=\psi_0(x_1,x_2),$$
$$(x_1,x_2)\in\Omega, \quad (6)$$
$$\xi(0,x_2,t)=0, \xi(l_1,x_2,t)=0, \frac{\partial\xi(0,x_2,t)}{\partial x_1}=0, \frac{\partial\xi(l_1,x_2,t)}{\partial x_1}=0$$
$$(x_2,t)\in(0,l_2)\times(0,T), \quad (7)$$
$$\xi(x_1,0,t)=0, \xi(x_1,l_2,t)=0, \frac{\partial\xi(x_1,0,t)}{\partial x_2}=0, \frac{\partial\xi(x_1,l_2,t)}{\partial x_2}=0,$$
$$(x_1,t)\in(0,l_1)\times(0,T).$$

Then, applying the formulas for integration by parts and using conditions (6), we hav
$$\int_Q\left(\frac{\partial^2\xi}{\partial t^2}+a^2\Delta^2\xi\right)u(\upsilon)dx_1dx_2dt=$$
$$=\int_\Omega u(x_1,x_2,T)\psi_0(x_1,x_2)dx_1dx_2$$
$$+\int_\Omega\psi_1(x_1,x_2)\frac{\partial u(x_1,x_2,T)}{\partial t}dx_1dx_2+$$
$$+a^2\left(\int_0^T\int_0^{l_2}\upsilon_1^0(x_2,t)\frac{\partial\Delta\xi(0,x_2,t)}{\partial x_1}dx_2dt\right.$$
$$-\int_0^T\int_0^{l_2}\upsilon_2^0(x_2,t)\frac{\partial\Delta\xi(l_1,x_2,t)}{\partial x_1}dx_2dt+$$
$$+\int_0^T\int_0^{l_2}\upsilon_1^1(x_2,t)\Delta\xi(0,x_2,t)dx_2dt-$$
$$\int_0^T\int_0^{l_2}\upsilon_2^1(x_2,t)\Delta\xi(l_1,x_2,t)dx_2dt+ \quad (8)$$
$$+\int_0^T\int_0^{l_1}\upsilon_3^0(x_1,t)\frac{\partial\Delta\xi(x_1,0,t)}{\partial x_2}dx_1dt$$
$$-\int_0^T\int_0^{l_1}\upsilon_4^0(x_1,t)\frac{\partial\Delta\xi(x_1,l_2,t)}{\partial x_2}dx_1dt+$$
$$+\int_0^T\int_0^{l_1}\upsilon_3^1(x_1,t)\Delta\xi(x_1,0,t)dx_1dt$$
$$\left.-\int_0^T\int_0^{l_2}\upsilon_4^1(x_1,t)\Delta\xi(x_1,l_2,t)dx_1dt\right)+$$
$$+\int_Q\xi\left(\frac{\partial^2u}{\partial t^2}+a^2\Delta^2u\right)u(\upsilon)dx_1dx_2dt.$$

Using (4), from (8) we obtain
$$a^2\left(\int_0^T\int_0^{l_2}\upsilon_1^0(x_2,t)\frac{\partial\Delta\xi(0,x_2,t)}{\partial x_1}dx_2dt\right.$$
$$-\int_0^T\int_0^{l_2}\upsilon_2^0(x_2,t)\frac{\partial\Delta\xi(l_1,x_2,t)}{\partial x_1}dx_2dt+$$
$$+\int_0^T\int_0^{l_2}\upsilon_1^1(x_2,t)\Delta\xi(0,x_2,t)dx_2dt$$
$$-\int_0^T\int_0^{l_2}\upsilon_2^1(x_2,t)\Delta\xi(l_1,x_2,t)dx_2dt+$$
$$+\int_0^T\int_0^{l_1}\upsilon_3^0(x_1,t)\frac{\partial\Delta\xi(x_1,0,t)}{\partial x_2}dx_1dt$$
$$-\int_0^T\int_0^{l_1}\upsilon_4^0(x_1,t)\frac{\partial\Delta\xi(x_1,l_2,t)}{\partial x_2}dx_1dt+$$
$$+\int_0^T\int_0^{l_1}\upsilon_3^1(x_1,t)\Delta\xi(x_1,0,t)dx_1dt-$$
$$\left.\int_0^T\int_0^{l_2}\upsilon_4^1(x_1,t)\Delta\xi(x_1,l_2,t)dx_1dt\right)=0,$$
$$\forall\upsilon_i^0,\upsilon_i^1\in L_2((0,l_2)\times(0,T)), i=1,2,$$
$$\forall\upsilon_i^0,\upsilon_i^1\in L_2((0,l_1)\times(0,T)), i=3,4.$$

Hence from arbitrariness
$$\upsilon_i^0,\upsilon_i^1\in L_2((0,l_2)\times(0,T)), i=1,2,$$
$$\upsilon_i^0,\upsilon_i^1\in L_2((0,l_1)\times(0,T)), i=3,4$$
we get that
$$\Delta\xi(0,x_2,t)=0, \Delta\xi(l_1,x_2,t)=0,$$
$$\frac{\partial\Delta\xi(0,x_2,t)}{\partial x_1}=0, \frac{\partial\Delta\xi(l_1,x_2,t)}{\partial x_1}=0,$$
$$(x_2,t)\in(0,l_2)\times(0,T), \quad (9)$$
$$\Delta\xi(x_1,0,t)=0, \Delta\xi(x_1,l_2,t)=0,$$
$$\frac{\partial\Delta\xi(x_1,0,t)}{\partial x_2}=0, \frac{\partial\Delta\xi(x_1,l_2,t)}{\partial x_2}=0,$$
$$(x_1,t)\in(0,l_1)\times(0,T).$$

Then from equation (5) and from conditions (7), (9), in view of the results from [15], it follows that
$$\xi(x_1,x_2,t)=0, (x_1,x_2,t)\in Q.$$

And from here it turns out
$$\psi_0(x_1,x_2)\equiv 0, \psi_1(x_1,x_2)\equiv 0$$
and the system is manageable.

The theorem has been proven.

Note that in order for all the above transformations to be legal, it would be necessary to first smooth out all the functions
$$\varphi_0(x_1,x_2)\in H^2(\Omega), \varphi_1(x_1,x_2)\in L_2(\Omega),$$
$$\upsilon_i^0,\upsilon_i^1\in L_2((0,l_2)\times(0,T)), i=1,2,$$
$$\upsilon_i^0,\upsilon_i^1\in L_2((0,l_1)\times(0,T)), i=3,4,$$
$$\psi_0(x_1,x_2)\in H_0^2(\Omega), \psi_1(x_1,x_2)\in L_2(\Omega)$$

carry out the indicated transformations for smooth solutions of the corresponding smoothed boundary value problems, and then go to the limit on the smoothing parameter and arrive at the required relations for weak solutions of boundary value problems [16]. We mean that when carrying out the above transformations, such a procedure has already been completed.

## IV. Conclusion

The work shows that the system under consideration (1)-(3) is controllable.

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