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Mathematical model of train movement

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Abstract. The problem of developing a high-quality and adequate mathematical model that describes the movement of a train using models of a viscoelastic material is proposed. A mathematical model that uses fractional derivatives more adequately describes the mechanical aspects of the movement of the object under consideration.

Keyword: *mathematical model; viscoelastic material; train; fractional derivatives*

1. INTRODUCTION

Description of the properties of many media, takes into account the loading mode and its "prehistory" and allows you to predict the behavior of these media under various loads. The calculation scheme of the railway train is presented as a multi-element complex of successively connecting viscoelastic bodies. The weight of the train during movement is also subjected to longitudinal and transverse deformations. The problem of developing a high-quality and adequate mathematical model that describes the movement of a train using models of a viscoelastic material is proposed. The model should not be too complex and allow the solvability of the corresponding problems.

2. PROBLEM STATEMENT AND METHOD SOLUTION

The calculation scheme of the railway train is presented as a multi-element complex of successively connecting viscoelastic bodies.

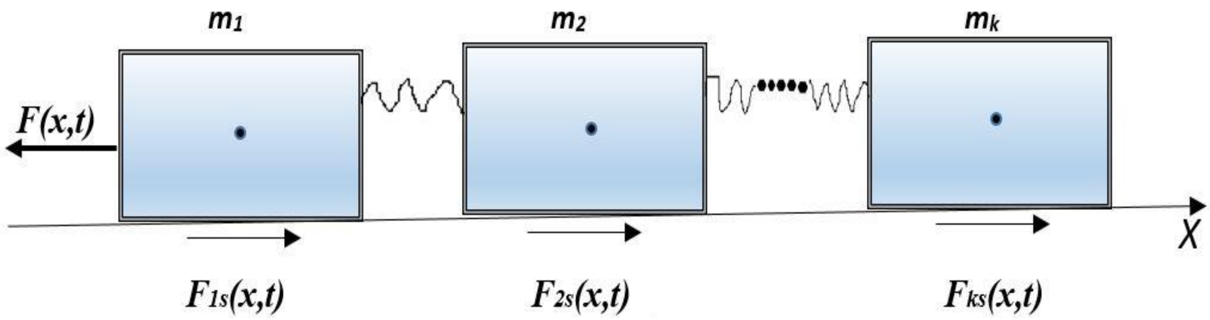

Fig 1. Calculation scheme of the train.

$$F(x,t) > \sum_{k=1}^{n} F_{sk}(x,t)$$

where *F(x,t)* is the electric locomotive traction force, m_i are masses of stock elements, $F_{is}(x,t)$ are friction forces.

2.1. Determination of the traction force of the electric drive and the forces of resistance to movement

The problem of transient processes of electric drives used in electric locomotives is investigated in order to study complex oscillations of the entire train during the motion process. At the same time, we consider the torsional vibrations faced by the train in the process of movement. The weight of the train during movement is also subjected to longitudinal and transverse deformations. These vibrations of the driving train can be described by nonlinear integral-differential equations. The movement of an AC electric drive is described by the equation [1]:

$$J\frac{d\omega_H(t)}{dt} = C_1\frac{2\Phi_{\partial\text{B}}\omega_P(t)}{1+\omega_P^2(t)} - M_H(t),$$

where ω_H – is the angular velocity of rotation of the rotor, J – is the moment of inertia of the actuator, ω_P – is the frequency of the current in the rotor, M_H – is the moment of resistance by the (load) of the actuator, C_1 – is a constant coefficient, $\Phi_{\partial\text{B}}$ – is the value of the useful resulting magnetic flux, and, in relative units has the following form

$$\frac{d\nu_H(t')}{dt'} = \frac{2\Phi^2\omega(t')}{1+\omega^2(t')} - \mu_H(t'), \tag{1}$$

where

$$\Phi = \frac{\Phi_{\partial\text{B}}}{\Phi_{kp}},\ \ \omega = \frac{\omega_P}{\omega_{kP}},\ \ \nu_H = \frac{\omega_H}{\omega_C},\ \ \mu_H = \frac{M_H}{M_{kP}},\ \ t' = \frac{t}{T_M'} \tag{2}$$

The derivation of the linear equation of the transverse vibrations of the trains with sequentially elastically connected wagons is explained in detail and the mechanics of the process is given in the papers [1,2,4]. The solution to this equation given in [2] is the function *F(x,t)*.

2.2. Selection of the viscoelasticity model of physically defining relations

In the standard model of a linear viscoelastic body, a series composed of time derivatives of the stress field is equated to a series composed of time derivatives of the strain field:

$$\sigma(t) + \sum_{i=1}^{m} b_i \frac{d^i}{dt^i}\sigma(t) = E_0\varepsilon(t) + \sum_{j=1}^{n} E_j \frac{d^j}{dt^j}\varepsilon(t) \tag{3}$$

This model is a generalization of the previously considered viscoelastic Voigt models. But for viscoelastic materials, the mechanical properties of which significantly depend on the frequency, the number of derivatives in these series is large. Therefore, working with the model takes a lot of time and leads to high-order differential equations [2,4], which greatly complicates the search for eigenvectors and eigenvalues.

The only model that can qualitatively match the results of experiments on polymeric materials is a model based on fractional derivatives. Fractional calculus has been successfully used to describe viscoelastic materials for almost 100 years.

2.3. Rationale for applying the fractional differential approach

The fractional differential approach has been successfully applied by many researchers to describe the rheological behavior of organic glasses, elastomers, polyurethane, and other materials. Depending on a number of factors (deformation conditions, sample geometry, etc.), the relationship between these two processes changes. For extremely oriented samples, the limiting process is the breaking of the main bonds, since the possibilities for the implementation of plastic deformation processes in them are small. In plastic polymers, where the volume of plastically (or forced-elastically) deformed material is large, a situation may arise when the energy required for plastic deformation and propagation of the main crack through the plastic deformation zone will be the limiting parameter of the fracture process. When a solid body is deformed, a heat flux caused by deformation is generated. According to the first law of thermodynamics

$$dU = dQ + dW$$

states that the change in internal energy in the sample (dU) is equal to the sum of the work (dW) done on the sample and the heat flux (dQ) into the sample. This relation is valid for any deformation, reversible or irreversible. There are two thermodynamic irreversible cases for which

$$dQ = -dW$$

These are the uniaxial deformation of the Newtonian fluid and the ideal elastoplastic deformation. For solid-state polymers, the deformation has a significantly different character: The Q/W ratio is not equal to a unit and varies within 0.35–0.75 depending on the nature of the material and the loading regime. In other words, thermodynamically ideal plasticity is not realized for these materials. The reason for this effect is the fractality of the structure of these polymers, which leads to fluidity not in the entire volume, but only in its part. The appearance of the fractional differentiation operator is due to the aforementioned fractality of the polymer structure.

Thus, the process of deformation of solid-state polymers is realized in a fractal space with the dimension d_f and is described using the fractal time t, which belongs to the points of the Cantor set. To describe evolutionary processes with fractal time, the mathematical apparatus of fractional differentiation and integration is used. In this case, the fractional exponent β coincides with the fractal dimension of the Cantor set and indicates the portion of system states that have been preserved over the entire evolutionary time t. For fractal objects in three-dimensional space, the parameter β is equal to the fractional part of d_f [3]:

$$\beta = d_f - 2.$$

The value of β characterizes the portion of the fractal (polymer structure) that does not change during deformation.

At present, the development of the concept of a fractal, using the mathematical apparatus of fractional integral-differentiation, has caused a trend to revise the basic principles of polymer mechanics. This helps to adequately describe systems with a complex spatial structure. Within its framework, it is possible to take into account the complex nature of nonlinear phenomena, such as memory effects and spatial correlations. In this case, not only previously known solutions are reproduced, but also their non-trivial generalization is given.

Another important feature is related to the self-similarity property of fractal structures. Unlike traditional ways of describing a system, the concept of a fractal takes into account the structure of the environment, thereby combining micro- and macroscopic levels of system description. It is this method that is important for complex multicomponent systems that are out of dynamic equilibrium, which are polymeric materials.

Also interesting is the case, which is a generalization of the behavior of a material with a single sudden change in the applied surface forces. Let us assume that a material with the properties of elasticity and creep described above is subjected to two non-simultaneously occurring changes in a uniform stress state, which are superimposed on one another. After the first application of stress, the behavior of the material will depend on time as well as on the magnitude of the initially applied stress. Let us now consider the situation that arises after an arbitrarily short time interval after the sudden application of the second stress. The behavior of the material will depend not only on the second change in external forces, but also on the continuing influence of the first applied stress.

Note that the behavior of an elastic material at any time depends only on the total stress level. Thus, the considered system of a more general type has a property that can be called the memory effect. Moreover, the behavior of this system is determined not only by the current stress state, but also by all past stress states, so that, generally speaking, the material "remembers" these past states. A similar situation arises if we turn to deformations; in this case, the current stress depends on the entire strain history. For this reason, some authors call viscoelasticity a hereditary theory.

It should be noted that although most of the achievements in the theory of viscoelasticity are recent, the theory formulated for the linear isotropic case has existed for a long time.

The founder of the use of fractional calculus in viscoelasticity, as noted, was P.G. Nutting, who in 1921 observed that the relationship between stress and strain for many complex materials is described by the equation [1]:

$$\sigma(t) = c_0 \gamma t^{-k},$$

where $0 < k < 1$.

At constant strain, the relationship yields an inverse power law of relaxation. An overview of some other work on the application of fractional calculus to viscoelasticity can be found in the relevant scientific and technical literature.

The calculation of the mechanical behavior of a moving train for a given time dependence of loading should be based on an adequate mathematical model. A mathematical model that uses fractional derivatives more adequately describes the mechanical properties of many media, takes into account the loading regime and its "prehistory", and makes it possible to predict the behavior of these media under various loads.

In general form, the model of physical relations of a viscoelastic structural element containing fractional derivatives has the form [1]:

$$\sigma(t) + \sum_{i=1}^{m} b_i D^{\alpha_i} \sigma(t) = E_0 \varepsilon(t) + \sum_{j=1}^{n} E_j D^{\beta_j} \varepsilon(t) \tag{4}$$

where $\sigma(t)$ is the stress, $\varepsilon(t)$ is the strain at time t, b_i, E_j, α_i, β_j – are model parameters, and D^{ρ} is the fractional differentiation operator of order ρ.

In general, the formulation of the equations of motion using fractional order derivatives requires setting no more than five empirical parameters. This is less than is usually required when using the standard model of a linear viscoelastic body (1), since this model is consistent with the basic physical laws of the phenomenon under consideration. Thus, the model based on fractional derivatives is attractive for engineering calculations. In most cases, the number of required parameters will be even less than five.

As shown in the scientific literature, to model many viscoelastic media, it is sufficient to restrict ourselves to the model

$$\sigma(t) + bD^{\alpha}\sigma(t) = E_0 \varepsilon(t) + E_1 D^{\beta} \varepsilon(t), \tag{5}$$

which, in the absence of instantaneous elasticity, which is typical for most polymers, is reduced to a simple model

$$\sigma(t) = E_1 D^{\beta} \varepsilon(t), \tag{6}$$

where $\sigma(t)$ is the stress, $\varepsilon(t)$ is the strain, E_1 and $0 < \beta < 1$ are the media material parameters. Here D^{β} – is the fractional differentiation operator in Caputo's definition.

The parameters of a viscoelastic medium are intermediate between the parameters of a viscous inelastic medium

$$(\beta = 1, \sigma(t) = E\varepsilon(t))$$

and an absolutely elastic nonviscous medium

$$\left(\beta = 0, \sigma(t) = \eta \frac{d\varepsilon(t)}{dt}\right),$$

i.e. the parameter β is a characteristic of the mechanical properties of the media material.

3. CONCLUSION

The proposed model is one of the first attempts to apply the fractional differential approach to solving such problems. The next task for the authors will be the solving of similar problems using the technique given in [1], when hereditary operators of defining relations are represented as well-studied tabulated Mittag-Leffler type functions.

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