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Mathematical Model of a Hexacopter-Type Unmanned Aerial Vehicle

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***Abstract*—At present, the East-West transport and communication corridor is in the center of attention of the world community as an international logistics project, and ensuring the security of its infrastructure is one of the important issues. The use of unmanned aerial vehicles as a means of operational control over the safe operation of both the railway infrastructure and the road system is considered to be very cost-effective. Due to its maneuverability, the use of hexocopters may be more appropriate among these types of devices. The article is devoted to the study of the dynamics of the Hexocopter type unmanned aerial vehicle (UAV), which can more effectively perform control functions. The motion of hexacopter is assumed as the motion of rigid body. The system of differential equations has been derived that represents the Cartesian coordinates of the orientation of hexacopter with respect to the inertial coordinate system and the relationship between quaternion based orientation of hexacopter and the rotational speed of its propellers. In order to illustrate the computer simulations of the flight these equations are discretized and solved by numerical methods. Computer simulations allow to determine the qualitative dependence of flight of the physical and technical parameters.**

Keywords—East-West transport and communication corridor, unmanned aerial vehicles, hexacopter, Newton-Euler equations, quaternions, system of differential equations, inertia tensor, inertial coordinate system, mathematical model

INTRODUCTION

Nowadays unmanned aerial vehicles are widely chosen in military and civil applications. Therefore, different types of unmanned aerial vehicles are created depending on the purpose of use [1, 2, 3]. Among the multicopters tricopter, quadrocopter and fila type unmanned aerial vehicles are very popular. The quadrotor type unmanned aerial vehicle increasingly attract the attention of researchers and widely manufactured among the multicopters. However, it is hardly controlled while one or more engine failures appear. Therefore, the multicopters with more than four rotors like hexacopters are reliable thanks to the stationary flight and high maneuverability.

Several type of unmanned aerial vehicles are created in Azerbaijan. The hexacopter type unmmanned aerial vehicles is called Əqrəb 5.0 (fig. 1). This article focuses on the simulation system of this hexacopter.

Fig. 1. General description of Əqrəb 5.0.

Hexacopter type unmanned aerial vehicle consist of 6 robotic arms. Each arm is equipped with rotor located on the vertices of hexagon. The axis of these rotors are parallel to the axis of symmetry passing through the center of mass of the hexacopter. The rotors create force and rotational torque. The hexacopter is controlled by adjusting the rotational frequencies of the rotors. In order to control the hexacopter by changing the rotational frequencies of the rotors it is necessary to write the formulas for the relationship between rotational frequencies and current orientation of hexacopter. The purpose of this article is to illustrate the mathematical model of the control of hexacopter type aircraft. The quadrotor type unmanned aerial vehicles with 4 rotors has been in the focus of researchers due to the stationary flight and stable hovering by balancing the forces that the four rotors produced. Nowadays, multicopters with more than 4 rotors are attracting the researcher attention like hexacopters and octocopters.

In order to test the characteristics of the flight and obtain the robust control of the unmanned aerial vehicle computer simulations are commonly illustrated.

Several type of unmanned aerial vehicles are created in Azerbaijan. In this work issues of creating simulation system are illustrated in order to experiment the flight of the hexacopter type unmanned aerial vehicle called Əqrəb 5.0.

MATHEMATICAL STATEMENT OF THE PROBLEM

As mentioned above, hexacopter consists of 6 robotic arms equipped with electric motors that are equidistant from the center of mass [5]. The rotors create force and rotational torque.

Let's numbering the rotors of the hexacopter as shown in the figure 2. It is assumed that the 1st, 3rd and 5th rotors are rotating clockwise, the 2nd, 4th and 6th rotors are rotating counterclockwise. The schematic illustration of the rotation are shown in the figure 2.

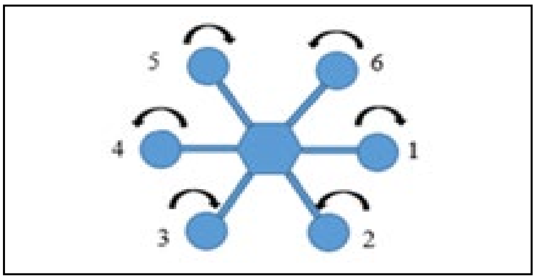

Fig. 2. The direction of rotation of the propellers of hexacopter.

In order to describe the equations of motion of the hexacopter two reference systems are necessary: the earth inertial coordinate system $OXYZ$ and body fixed coordinate system $oxyz$.

Suppose that, the origin of the inertial frame is fixed in a point located on the surface of earth. The OY axis of the inertial frame is directed to the North, OX is directed to the East and OZ axis is directed upwards perpendicular to the OXY plane.

Assume that, Ox axis of the body fixed frame is directed towards the 1st robotic arm of the hexacopter, Oy axis is perpendicular to Ox axis and Oz is directed upwards perpendicular to Oxy plane along the line of symmetry of hexacopter (fig 3).

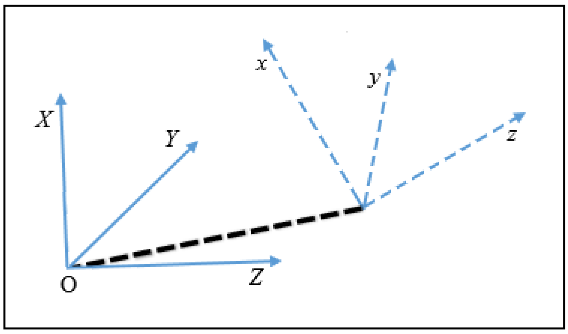

Fig. 3. Inertial and body fixed coordinat systems

As is obviously known that, in order to describe the rotation of an unmanned aerial vehicle Euler angles are used [7]. The orientation of the body fixed frame with respect to the earth inertial frame is usually described by the means of Euler angles. If these angles are known, then any vector in the body fixed frame can be transformed to the inertial frame. The transformation from the body fixed frame to the earth inertial frame is obtained by using ration matrix R [11].

However, the description of the orientation of hexacopter by means of Euler angles suffer from singularities at certain orientations. For overcoming these problems, a new parametrization, the quaternions, are commonly used [4]. Here is a brief information about the quaternion based rotation of aircraft.

It is known that, in theoretical mechanics the rotation of rigid body is described by rotation vector [7]. The rotation vector is characterized by the rotation axis and angle of rotation. If α is angle of rotation about the unit vector (u_x, u_y, u_z), then the rotation vector is $u = \alpha \cdot (u_x, u_y, u_z)$, here $u_x^2 + u_y^2 + u_z^2 = 1$. As is clearly seen that, the rotation angle is represented by means of 4 quantities α, u_x, u_y, u_z.

Quaternions are generally represented in the form of $q_0 + q_1 i + q_2 j + q_3 k$ where q_0, q_1, q_2, q_3 are real numbers, i, j, k are imaginary units. They form a four dimentional associative division algebra denoted by H(Hamilton) and satisfy the following terms:

$$i^2 = j^2 = k^2,$$

$$ij = k,\ jk = i,\ ki = j,$$

$$ji = -k,\ kj = -i,\ ik = -j.$$

The characteristics of quaternions allow them to be used in order to express the angles of rotation. The rotation vector u is represented as follows via quaternion formulation:

$$q_0 = \cos\frac{\alpha}{2},$$

$$q_1 = u_1 \sin\frac{\alpha}{2},$$

$$q_2 = u_2 \sin\frac{\alpha}{2},$$

$$q_3 = u_3 \sin\frac{\alpha}{2}.$$

The transformation of any vector from the inertial coordinate system to the body fixed coordinate system can be expressed by means of the transformation matrix Q:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}. \quad (1)$$

The following hypotheses and agreements are taken into account:

- It is assumed that during the translational motion the shape of the hexacopter does not play an important role. Therefore, it can be assumed as rigid body.
- The gyroscopic forces created by the rotation of hexacopter can be ignored due to the small angular velocities
- Gravity and aerodynamic forces do not create a torque, because they affect the center of mass of hexacopter
- It is assumed that, the constructive parameters of hexacopter are illustrated as shown in the figure 4 and the mass of the vehicle is distributed only along its 6 robotic arms and the gravity center of the central block.

At the considered instant t, the current coordinates of hexacopter relative to $OXYZ$ coordinate system are denoted by $X(t), Y(t), Z(t)$, the components of the quaternion that represent the transformation from the body fixed coordinate system to the inertial coordinate system are denoted by $q_0(t) + q_1(t)i + q_2(t)j + q_3(t)k$ and the rotational frequency of the propellers of arms are represented by $\omega_i,\ \ i = 1, \dots, 6$.

The setting a mathematical relationship between the quantities $x(t)$, $y(t)$, $z(t)$, $q_0(t)$, $q_1(t)$, $q_2(t)$, $q_3(t)$ and the rotational frequency of propellers $\omega_1, \omega_2, \dots, \omega_6$ is the mathemamtical model of hexacopter.

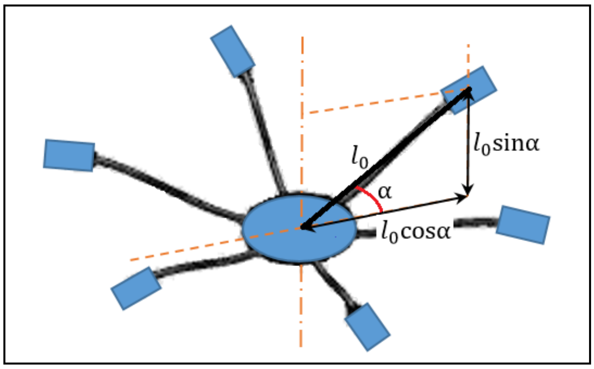

Fig. 4. Constructive dimensions of the hexacopter

SOLUTION OF THE PROBLEM

First of all, we need to represent the equations of translation. Assume that, the mass of the rotors of hexacopter centered at the gravity are denoted by m_6, the mass of the central block in the center of gravity of drone is denoted by m_0. According to the Newton's second law of motion in the inertial frame the force acting on the object is equal to the multiplication of acceleration and mass of the object. Taking into account the hypothesis and agreements mentioned above, let's calculate the forces acting on the hexacopter:

Gravitational force acting on the hexacopter.

$$\boldsymbol{F}_G = m \cdot (0, 0, -g)^T, \qquad (2)$$

where $m = m_0 + 6m_*$ is the total mass of hexacopter, $g = 9.81\ \frac{m}{san^2}$ is gravitational constant and T is sign of transpose.

Thrust force of the rotors.

The main force affecting the movement of hexacopter is thrust force created by the propellers and rotors that lift the unmanned aerial vehicle in the air. As shown in the construction of the hexacopter all rotors of hexacopter generate a force towards the vector $(0, 0, 1)$. Therefore, the trust force generated by every rotor i is represented as follow:

$$\boldsymbol{f}_i = (0, 0, k\omega_i^2)^T.$$

Then the total thrust force is denoted by:

$$\boldsymbol{f}_M = \sum_{i=1}^{6} \boldsymbol{f}_i = k(0, 0, \sum_{i=1}^{6} \omega_i^2)^T.$$

The thrust force $\boldsymbol{F}_M$ generated by the motors with respect to the inertial coordinate system $OXYZ$ can be obtained by applying the rotation matrix Q on the force f_M at the considered instant $t \geq 0$.

$$\boldsymbol{F}_M = \boldsymbol{Q}^{-1} \cdot \boldsymbol{f}_M = k \cdot \boldsymbol{Q} \cdot (0, 0, \sum_{i=1}^{6} \omega_i^2)^T. \qquad (3)$$

Aerodynamic forces acting on hexacopter.

The velocity of motion of the hexacopter with respect to the inertial coordinate system is represented by the derivatives of its current coordinates: $(X'(t), Y'(t), Z'(t))$.

If the velocity of the vehicle with respect to the body fixed coordinate frame is denoted by $(v_x(t), v_y(t), v_z(t))$, then the following equation is derived:

$$\boldsymbol{(v_x(t), v_y(t), v_z(t))^T = Q \cdot (X'(t), Y'(t), Z'(t))^T}.$$

As is obvious that the drag force acting on the hexacopter is opposite to the direction of movement of rigid body. Therefore, this force is denoted by the following formula with respect to the body fixed coordinate system:

$$f_A = -c_A \left(v_x(t), v_y(t), v_z(t)\right)^T \left|\left(v_x(t), v_y(t), v_z(t)\right)\right| =$$

$$= -c_A \boldsymbol{Q}^{-1} \left(X'(t), Y'(t), Z'(t)\right)^T \times \left|\left(X'(t), Y'(t), Z'(t)\right)\right|.$$

Then the aerodynamic drag force with respect to the $OXYZ$ coordinate system can be obtained as follows:

$$\boldsymbol{F_A} = \boldsymbol{Q^{-1}} \cdot \boldsymbol{f_A} = -c_A \boldsymbol{Q} \left[\boldsymbol{Q}^{-1} \left(X'(t), Y'(t), Z'(t)\right)^T \times\right.$$

$$\left. \times \left|\left(X'(t), Y'(t), Z'(t)\right)\right|\right]. \qquad (4)$$

Taking into consideration that, the acceleration of the motion is denoted by $(X''(t), Y''(t), Z''(t))$, then according to the Newton's second law the equation of motion of hexacopter is derived as follows:

$$m(X''(t), Y''(t), Z''(t))^T = \boldsymbol{F}_G + \boldsymbol{F}_M + \boldsymbol{F}_A,$$

$$m\left(X''(t), Y''(t), Z''(t)\right)^T =$$

$$= k(0, 0, \sum_{i=1}^{6} \omega_i^2)^T + k\boldsymbol{Q} \cdot (0, 0, \sum_{i=1}^{6} \omega_i^2)^T -$$

$$-c_A \boldsymbol{Q} \times \left[\boldsymbol{Q}^{-1} \cdot \left(X'(t), Y'(t), Z'(t)\right)^T \times\right.$$

$$\left. \times \left|\left(X'(t), Y'(t), Z'(t)\right)\right|\right]. \qquad (5)$$

In order to derive the equation of rotational motion of hexacopter, it is necessary to calculate the elements of inertial tensor of hexacopter:

$$\boldsymbol{J} = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix}$$

Let's denote the distance between the gravity center of the central block and gravity center of robotic arms of hexacopter by l_0, and α is the angle between the line connecting these 2 points with the plane that central block of the hexacopter located.

According to the figure 4, the elements of the inertial tensor $\boldsymbol{J}$ are obtained:

$$J_{xx} = \sum_{k=1}^{6} m_k (y_k^2 + z_k^2) = 3m_* l_0^2 (1 + \sin^2),$$

$$J_{yy} = \sum_{k=1}^{6} m_k(x_k^2 + z_k^2) = 3m_* l_0^2(1 + \sin^2),$$

$$J_{zz} = \sum_{k=1}^{6} m_k(x_k^2 + y_k^2) = 6m_* l_0^2 \cos^2,$$

$$J_{xy} = J_{yx} = \sum_{k=1}^{6} m_k x_k y_k = 0,$$

$$J_{xz} = J_{zx} = \sum_{k=1}^{6} m_k\, x_k z_k = 0,$$

$$J_{xz} = J_{zx} = \sum_{k=1}^{6} m_k\, x_k z_k = 0.$$

Thus,

$$\boldsymbol{J} = \begin{pmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{pmatrix} = diag\left(J_{xx}, J_{yy}, J_{zz}\right).$$

Let's denote the angular rotational angular velocity of hexacopter by $\boldsymbol{w}(t) = \left(w_x(t), w_y(t), w_z(t)\right)$. Then, the following equation can be represented on the basis of total torque M generated by the forces acting on the vehicle [8,9]:

$$\boldsymbol{J w' + w \times (J w) = M}. \qquad (6)$$

Or, it can be indicated via components:

$$J_1 w_1' + (J_3 - J_2) w_2 w_3 = M_1$$

$$J_2 w_2' + (J_1 - J_3) w_1 w_3 = M_2$$

$$J_3 w_3' + (J_2 - J_1) w_1 w_2 = M_3$$

According to the agreements mentioned above gravity and aerodynamic forces do not create a torque, the torque M can be derived by the sum of the torques generated by the rotation of propellers.

The sum of the rotation torques created by the rotation of first, third and fifth propellers is $\sum_{i=1,3,5} b\omega_i^2$ and the torque generated by second, fourth and sixth propellers is denoted by $\sum_{\boldsymbol{i=2,4,6}}\left(\boldsymbol{-b\omega_i^2}\right)$. Therefore, total yaw moment would be derived by $M^{(Oz)} = b\left(\sum_{i=1,3,5} \omega_i^2 - \sum_{i=2,4,6} \omega_i^2\right)$.

From the geometrical structure of hexacopter represented in the figure 4, it is possible to get information about roll and pitch moments:

$$M^{(Ox)} = -\frac{\sqrt{3}}{2} k l_0 \omega_2^2 \cos\alpha + \frac{\sqrt{3}}{2} k l_0 \omega_3^2 \cos\alpha +$$

$$+\frac{\sqrt{3}}{2} k l_0 \omega_5^2 \cos\alpha - \frac{\sqrt{3}}{2} k l_0 \omega_6^2 \cos\alpha =$$

$$= \frac{\sqrt{3}}{2} k l_0 \cos\alpha\, (-\omega_2^2 + \omega_3^2 + \omega_5^2 - \omega_6^2),$$

$$M^{(Oy)} = k l_0 \omega_1^2 \cos\alpha - \frac{1}{2} k l_0 \omega_2^2 \cos\alpha + k l_0 \omega_3^2 \cos\alpha -$$

$$-k l_0 \omega_4^2 \cos\alpha + \frac{1}{2} k l_0 \omega_5^2 \cos\alpha - k l_0 \omega_6^2 \cos\alpha =$$

$$= kl \cos\alpha \left(\omega_1^2 - \frac{1}{2}\omega_2^2 + \omega_3^2 - \omega_4^2 + \frac{1}{2}\omega_5^2 - \omega_6^2\right).$$

Thus,

$$\boldsymbol{M} = \begin{bmatrix} \frac{\sqrt{3}}{2} k l_0 \cos\alpha\, (-\omega_2^2 + \omega_3^2 + \omega_5^2 - \omega_6^2) \\ k l_0 \cos\alpha \left(\omega_1^2 - \frac{1}{2}\omega_2^2 + \omega_3^2 - \omega_4^2 + \frac{1}{2}\omega_5^2 - \omega_6^2\right) \\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2 + \omega_5^2 - \omega_6^2) \end{bmatrix}.$$

Taking into consideration the angular velocity $\boldsymbol{w}(t)$ deriven from the equations (6), it is possible to calculate the quaternion form of the current orientation of hexacopter solving the following differential equations:

$$\begin{aligned} q_0' &= \frac{1}{2}(-q_1 w_1 - q_2 w_2 - q_3 w_3), \\ q_1' &= \frac{1}{2}(q_0 w_1 + q_2 w_3 - q_3 w_2), \\ q_2' &= \frac{1}{2}(q_3 w_1 + q_0 w_2 - q_1 w_3), \\ q_3' &= \frac{1}{2}(-q_2 w_1 + q_1 w_2 + q_0 w_3). \end{aligned} \qquad (7)$$

As is clearly seen, the system (5) – (7) is the system of ordinary differential equations with respect to $X''(t)$, $Y''(t)$, $Z''(t)$, $w_x(t)$, $w_y(t)$, $w_z(t)$, and $q_0(t)$, $q_1(t)$, $q_2(t)$, $q_3(t)$. In order to solve the equations, lets represent the initial conditions:

$$\begin{cases} X(t)|_{t-0} = X_0, \\ Y(t)|_{t-0} = Y_0, \\ Z(t)|_{t-0} = Z_0, \end{cases} \qquad (8)$$

$$\begin{cases} X'(t)|_{t-0} = \acute{X}_0, \\ Y'(t)|_{t-0} = \acute{Y}_0, \\ Z'(t)|_{t-0} = \acute{Z}_0, \end{cases} \qquad (9)$$

$$\begin{cases} w_x(t)|_{t-0} = w_{x0}, \\ w_y(t)\big|_{t-0} = w_{y0}, \\ w_z(t)|_{t-0} = w_{z0}, \end{cases} \qquad (10)$$

$$\begin{cases} q_0(t)|_{t-0} = q_{00}, \\ q_1(t)|_{t-0} = q_{10}, \\ q_2(t)|_{t-0} = q_{20}, \\ q_3(t)|_{t-0} = q_{30}. \end{cases} \qquad (11)$$

Thus, the system (4) – (11) represents the equation of motion of the hexacopter. On the basis of angular rotational velocity ω_i $(i = 1,2,\ldots,6)$ of propellers of rotors of the hexacopter this system allows to calculate the trajectory and orientation of the vehicle.

In order to generate the controlled computer model of the hexacopter the discrete analog of the system (4) – (11) is indicated. On the basis of this model, computer simulations of the flight make it possible to determine the dependence of the quality of the hexacopter flight on the physical and technical parameters.

CONCLUSION

The “East-West Corridor” project is a global energy and transport project that has solved a number of problems in various fields of science. The use of hexacopter type unmanned aerial vehicle is considered more convenient in terms of more efficient control of transport infrastructure. To create an appropriate vehicle the system of differential equations of the Cartesian coordinates of the orientation of hexacopter with respect to the inertial coordinate system and the relationship between the orientation represented via quaternions and the angular velocity of the propellers of hexacopter rotors are generated. In order to put into practice, the computer simulations of the flight, these equations are discretized and solved numerically. The computer simulation permits to determine the quality dependence of the flight on the physical and technical parameters.

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