On Two Problems of Combinatorial Optimization in the Analysis of Road Networks

Aleksandr Soldatenko
Institute of Mathematics and Computer Sciences
Siberian Federal University
Krasnoyarsk, Russia
ASoldatenko@sfu-kras.ru

Daria Semenova
Institute of Mathematics and Computer Sciences
Siberian Federal University,
Krasnoyarsk State Medical University
Krasnoyarsk, Russia
DVSemenova@sfu-kras.ru

***Abstract*— Two combinatorial optimization problems for the road networks analysis are considered in the paper. The shortest path problem is a traditional graph theory problem. Extensions of this problem have many real applications, for example, for modeling road or telecommunication networks. Thus, the shortest path problem in a non-stationary metric network allows to model road situations in the traffic congestion appearance conditions. The shortest path finding in the resource-limited network allows to display situations related to limited fuel or time reserve. Another problem is to find all the maximum induced bicliques in the network. This problem provides information about the structure of the network. The paper presents an analysis of real road networks based on the results of computational experiments. The real road networks analysis based on computational experiments results is presented in the paper.**

Keywords—road networks, routing algorithms, maximal bicliques, network analysis

I. Introduction

With the development of urban infrastructure a need for the road networks analysis, design and restructuring appears. The use of graph and hypergraph models in the road networks analysis is one of the topical areas of modern scientific research. At the same time, for modeling and analyzing the functioning of such networks, it is necessary to solve the combinatorial optimization problems of finding and enumerating various configurations. Of particular interest are configurations such as shortest paths and maximum bicliques, since they allows to determine the network bottlenecks, the main highways, and perform preprocessing to divide the original network into subnets.

Currently, the solution of logistics problems is relevant both for a separate district, city, region and for the whole country. There are many algorithms for solving the routing problem without additional restrictions on the network. However, in case of additional restrictions imposed on the network, the classical algorithms do not applicable for the shortest paths constructing. In addition, an increase of the analyzed road networks dimensionality significantly affects the performance of the using algorithms. This is due to the fact that classical algorithms are either unable to find an exact solution to the problems posed, or find it in an unacceptable time. To analyze the road network structure, one can use the problem of enumerating all the maximum induced bicliques. Let's note that this problem is an intractable task, since the number of bicliques can exponentially depend on the size of the original network. Thus, not only new algorithms are in demand for solving problems of the road networks analysis, routing and design, but also the approaches increase the existing algorithms performance.

The aim of the work is the road networks analyze of Krasnoyarsk, Tomsk, Novosibirsk and Baku cities. The analysis is given in detail on the example of the city Krasnoyarsk. The analysis is carried out using three algorithms developed by the authors to solve the problems of finding the shortest path with constraints and enumerating all the maximum induced bicliques [1]–[3].

II. Statements of Problems

In the article, the analysis of road networks is carried out on the basis of solving the following problems: determining the shortest path with additional restrictions on the network and finding all the maximum induced bicliques.

The Table I shows the notation and definitions of the graph and hypergraph theory used in the article according to the works [1]–[5].

The finding shortest path classical problem assumes finding the least weight route in a weighted graph. Wherein the graph can be either undirected or directed.

Shortest Path problem (SP). A weighted directed graph $G = (V, E)$ with weight function w_e, where $e \in E$ is arc and shortest path (s, d)-query are given. It is required to find for the (s, d)-query the path P of the least weight $\sum_P w_e$ and the sequence of vertices that form it.

This problem has been well studied and a large number of algorithms have been developed for it [4]–[6]. One of the classical algorithms is Dijkstra's algorithm [5]. In such statement SP problem is polynomial solvable [4]. However, in most cases, when additional constraints or requirements are imposed on the shortest path or input graph, the problem becomes much more complicated and belongs to the *NP* class.

We consider two extensions of SP problem: Time-Dependent Shortest Path problem for the metric network that satisfies FIFO condition and Resource Constrained Shortest Path problem.

A. Time-Dependent Shortest Path problem

A time-dependent network is a graph whose edge weights can change over time. So, weight of any edge is defined as function of time (Table I, line 6). This approach allows to simulate traffic congestion at different times of the day.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement No. 075-02-2020-1534/1).

TABLE I. DEFINITIONS AND DESIGNATIONS

1	$G = (V, E)$	Directed graph without loops and dual edges, where V is a set of vertices and E is a set of edges while $\lvert V\rvert = n$ and $\lvert E\rvert = m$
2	$t \in T$	A value t is measured in conventional units of time and takes values from a finite set T
3	w_e	Weight function for an edge $e \in E$
4	$r_i(e)$	Resource functions for an edge $e \in E$ where $i = \overline{1,k}$
5	R_i	Positive values reflecting the resource capabilities of the network where $i = \overline{1,k}$
6	$w_e(t)$	weight function depending on time t for an edge $e \in E$
7	$F_e(t) = t + w_e(t)$	Function of arrival at the vertex y when moving along an edge $(x, y) = e$
8	$F_e(t_1) \le F_e(t_2)$	FIFO condition (monotonicity condition) for any moments of time $0 < t_1 \le t_2$
9	P	Path P from vertex s to vertex d
10	$w(P) = \sum_{e \in P} w_e$	Weight of the path P
11	$w(P, t_s) = t_s + \sum_{e \in P} w_e(t_i)$	Weight of the path (P, t_s) where $t_0 = t_s$, $t_i+1 = F_e(t_i)$
12	$dist(s, d, t_s) = \min_P \left\{ w(P, t_s) : s \xrightarrow[P]{} d \right\}$	Weight of shortest (s, d, t_s)-path
13	Feasible (s, d)-path P	Path P which satisfies $\sum_{e \in P} r_i(e) \le R_i$ where $i = \overline{1,k}$
14	Optimal (s, d)-route	Feasible (s, d)-path P with minimal weight $w(P)$
15	Bipartite graph	Graph G with parts S_0 and S_1 which satisfies that $x \in S_0$ and $y \in S_1$ for any edge $e = (x, y) \in E$
16	Complete bipartite graph (a biclique)	Bipartite graph that contains all possible edges
17	Maximal biclique	biclique that cannot extended with additional adjacent vertices

Time-Dependent Shortest Path problem (TDSP). A time-dependent network $G = (V, E)$ and (s, d, t_s)-query are given. It is required to find value $dist(s, d, t_s)$ for the (s, d, t_s) shortest path and the sequence of vertices that form it.

The problem in such statement belongs to the class of *NP*-hard problems. However, if network is metric and satisfies FIFO condition (Table I, line 8) then TDSP problem is polynomial solvable. The example of the weight function $w_e(t)$ for the edge e of the time-dependent metric network satisfying FIFO condition is shown at fig. 1. Such TDSP problem formulation is considered in this work.

Many algorithms are known for finding the shortest paths in a graph [5]–[6], some of them are adapted for time-dependent networks [7]–[10]. The classical Dijkstra's algorithm is able to solve the problem of finding the shortest path in a time-dependent metric network with FIFO condition in $O(n^2)$ time. In developing modern routing systems, one has to deal with the network dimensionality problem. Thus, a network can contain several hundreds of thousands of nodes, and it is required to find the optimal route between the nodes of this network in a few seconds. Under such conditions, classical algorithms for finding the shortest paths, including Dijkstra's algorithm, work for an unacceptably long time. To cope with this problem, various methods of speeding up classical algorithms are used [5]. Most of these techniques are based on speeding up Dijkstra's algorithm by applying a two-phase approach to solving the TDSP problem. Various implementations of the two-phase approach of Dijkstra's algorithm are known [10]. They are usually divided into the following main groups: hierarchical algorithms (algorithms based on a multilevel representation of the original graph) [7], labeling algorithms [9], landmark routing algorithms [11]–[12]. Note that most of these modifications do not exceed Dijkstra's algorithm for time-dependent metric networks in theoretical complexity.

B. *Resource Constrained Shortest Path problem*

A resource-constrained network assumes that each edge in the graph has multiple weights. One is considered the main weight, over which the minimization is carried out, in the process of determining the shortest path. Other weights define different resources. Thus, the network or request contains the resource possibilities for searching the shortest path. This problem allows to simulate various traffic situations associated with toll highways, limited fuel or road capacity. Another extension of the shortest path problem assumes that each edge in the graph has multiple weights, and is formulated as follows.

Resource Constrained Shortest Path problem (RCSP). A directed graph $G = (V, E)$ on the edges of which positive real-valued functions are defined w_e, $r_i(e)$ and values R_i, $i = \overline{1,k}$ are given. It is required to find (s, d)-route and the sequence of vertices that form it.

It is known that even with one additional resource the RCSP problem is *NP*-hard [4], [13]. The resource is represented by an additional weighting function and a value that is a resource constraint (Table I, line 4-5). Example of such network is shown at fig. 2. The RCSP problem statement is admissible in terms of linear integer programming. Thus, at present, there are three classes of methods and corresponding

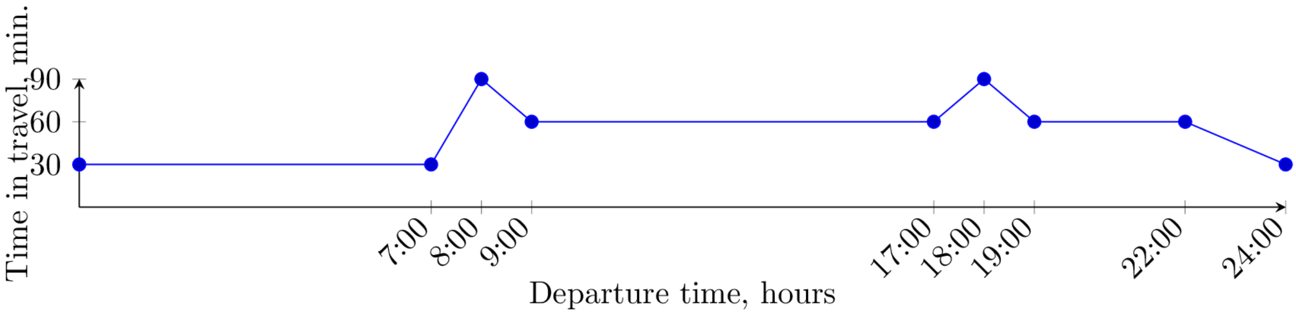

Fig. 1. Weight function of an edge that satisfies FIFO condition

algorithms capable of finding an exact or approximate solution to the RCSP problem: path ranking methods [14], vertex marking methods [15]–[18], Lagrangian relaxation methods [19]–[21]. The first two classes of methods are based on the graph-theoretical formulation of the problem, while the methods of the third class are based on the formulation of the RCSP problem in an integer linear programming language. Most of the algorithms based on the graph- theoretical formulation of the problem are extensions of Dijkstra's algorithm or similar ones. [15]–[16], [20].

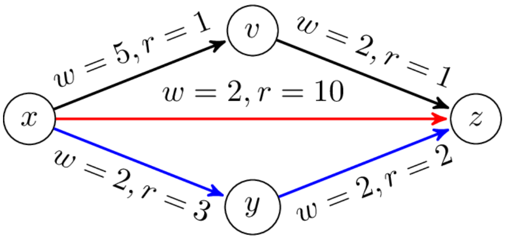

Fig. 2. Network with resource constrain R = 5 where red path does not feasible, black is a feasible and blue is the optimal route

Path ranking methods involve solving the k-shortest path problem in a graph. In this problem, it is required to find not only the shortest path, but also the next k–1 shortest paths, which may differ from the shortest and be heavier in the sense of the weight function. Since the optimal path for the RCSP problem may differ from the shortest path, in the sense of the resource function, the parameter k cannot be initially determined. Thus, the approach of finding k-shortest paths becomes more complicated by going through all the shortest paths until a feasible one is found. A prominent representative of this direction is Jen's algorithm, which finds a solution to the k-shortest paths problem in time $O(k \cdot n \cdot (m + n \cdot \log n))$, where k can exponentially depend on the size of the original network [14].

Vertex labeling methods involve the use of additional labels for correct routing in a resource-constrained network. This class of methods contains both exact algorithms for solving and approximate ones. Approximate algorithms are divided into two types: approximate in weight function, approximate in resource functions. The exact solution algorithm has complexity $O(N^5 \cdot b \cdot \log (n \cdot b))$ where b is the greatest weight of an edge in the graph [15]. The execution time of this algorithm is high. Therefore, studies of approximate algorithms for solving the problem are in demand. The most efficient algorithm for an approximate solution was proposed in [22] and has computational complexity $O(m \cdot n \cdot (1/\varepsilon + \log \log n))$. The algorithm finds a feasible path different from the optimal one at most $(1 + \varepsilon)$ times. Algorithms that are approximate in terms of resource functions are widely used in telecommunication networks. The most efficient algorithm is presented in [23] and has the following complexity $O(n \cdot m \cdot \log \log \log n + m \cdot (n/\varepsilon)^{k-1})$ where k is the number of resource constraints. It finds the shortest path that exceeds the resource constraints by no more than $(1 + \varepsilon)$ times.

Lagrangian relaxation techniques mainly consist of three phases. In the first phase, the lower and upper bounds of the optimal values for the problem are calculated. In the second phase, these boundaries are used to simplify the original graph. In the third phase, the gap between the boundaries is closed by finding the optimal route. For this purposes various algorithms for finding the shortest paths are used. Algorithms in this direction are presented in the works [19]–[21], [24].

C. Maximal Induced Bicliques Generation problem

The problem of finding all the maximal induced bicliques is formulated as follows.

Maximal Induced Bicliques Generation problem (MIBG). Let the hypergraph $G = (V, E)$ without double edges is given. It is necessary to find a set of all maximal induced bicliques.

It is known that complexity of the MIBG problem does not be easier than the NP-hard problem, since the problem of finding one maximal induced biclique is NP-hard [25]–[26]. A number of graph-theoretic problems that belong to the class of #P-complete or NP-complete problems are reduced to the search for bicliques [4]. Note that in the general case the number of maximal bicliques exponentially depends on the size of the graph [27].

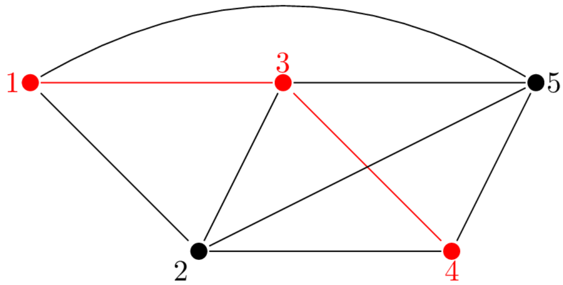

Fig. 3. Maximal induced biclique with parts $S_0 = \{1, 4\}$ and $S_1 = \{3\}$

The problem of finding all maximal bicliques is being actively studied for networks represented by a graph. There are two areas of research, the search for all maximal non-induced bicliques and maximal induced bicliques. Bicliques $X' = S_0 \cup S_1$ are called induced then each one of the sets S_0 and S_1 are independent, i.e. vertices in a set are not adjacent. If property of independency does not have place then bicliques are called non-induced. Example of the maximal induced biclique is shown at fig. 3. Algorithm for finding all maximal non-induced bicliques with the complexity $O(a^3 \cdot 2^{2a} \cdot n)$ where a is the arboricity of the input graph [28]. A number of researchers note that in practice, only maximal non-iduced bicliques of large dimensionality are in demand [29]–[30]. In the paper [29] the algorithm for the maximal non-induced bicliques search with parameter p for biclique size threshold has been proposed. The complexity of this algorithm is $O(n \cdot m \cdot N)$ where N is the set of all non-induced bicliques with size greater or equal to p. For finding all maximal induced bicliques of graph in the paper [28] algorithm is proposed. It has complexity $O(n \cdot k \cdot (\Delta + k) \cdot 3^{(\Delta + k)/3})$, where Δ is maximum degree of a vertex and k is degeneracy of graph.

Table II presents three algorithms and estimations of their complexity on time to solve the considered problems. Theoretical studies and assessment complexity of the algorithms presented in the works [1]–[3].

For the problem of the shortest path in a time-dependent metric network with the FIFO condition, the modified ALT algorithm is used [1]. The algorithm is based on a specific representation of the edges of the original network. For such a representation, a sufficient condition for the continuity and admissibility of potential functions is shown in paper [1], which are necessary for the correct operation of the ALT algorithm.

TABLE II. THE PROBLEMS CONSIDERED AND THE ALGORITHMS FOR THEIR SOLUTION

Problem		Algorithm	
Name	***Complexity class***	***Name***	***Complexity***
TDSP	*P*	ALT	$O(n^2)$
RCSP	*NP*-hard	RevTree	$O(n^2)$
MIBG	At least *NP*-hard	HFindMIB	$O(2^{2\Delta} \cdot \Delta \cdot (\lvert MBC\rvert + \Delta^3 \cdot \log(2^{2\Delta})) + n^2)$

To find the shortest path in a resource constrained network with one resource, the RevTree algorithm was developed earlier [2]. A distinctive feature of the proposed algorithm is that the accuracy of the solution can be estimated based on the weight and resource functions of the original network.

The search for all maximum induced bicliques is implemented by the HFindMIB algorithm [3]. The algorithm differs from previously existing algorithms in that it uses a hypergraph approach in bicliques generation process. The algorithm finds and lists in lexicographic order all maximal induced bicliques in graph and hypergraph networks.

III. COMPUTATIONAL EXPERIMENTS AND NETWORK ANALYSIS

Computational experiments were carried out for the modified ALT algorithm, RevTree and HFindMIB algorithms. Based on the results obtained, the analysis of the road networks under consideration is carried out. The experiments were carried out on real road networks from the DIMACS database [31], as well as on road networks obtained from the OpenStreetMap web-mapping project using the OSMnx package for Python [32]. Computational experiments were performed on a PC with an AMD Ryzen 5 3600 6-Core Processor 3.60 GHz and 16 GB of RAM. For the modified ALT algorithm and RevTree algorithm, 1000 queries were generated for each network. The HFindMIB algorithm was applied to undirected versions of the presented networks.

The first series of computational experiments was carried out for the modified ALT algorithm. The algorithm periodically sets up landmarks that it uses in its work. According to the results of work [33], the number of landmarks was assumed to be equal to 12, and the update period was equal to 30. The results are presented in Table III.

TABLE III. RESULTS FOR THE MODIFIED ALT ALGORITHM

Graph	***n***	***m***	**Time, sec.**
TOF	3840	9639	8.151
KJA	4362	10536	9.030
NSK	7357	19106	27.838
GYD	12458	28936	69.052

Heat map for KJA network is presented at fig. 4. A road section is red then it is most required in shortest paths and blue otherwise. Edges that have not met in any of the shortest paths are marked in black.

The second series of experiments was carried out for the RevTree algorithm. The RCSP problem do not always have a solution, since a feasible path may not exist for the request. The request is considered to have been complete if a solution has been found for it. The results are shown in Table IV.

TABLE IV. RESULTS FOR THE REVTREE ALGORITHM

Graph	***n***	***m***	**Time, sec.**	**Percentage of completed requests**
TOF	3840	9639	1.708	0.483
KJA	4362	10536	7.286	0.849
NSK	7357	19106	12.212	0.680
GYD	12458	28936	15.998	0.539

A similar heat map is presented based on the results of solving the RCSP problem for the KJA network in fig. 5.

It can be seen that the heat maps in fig. 4 and 5 have common road sections. This suggests that these roads are equally in demand in both time-dependent and resource constrained networks. Thus, these roads are main for considered network.

The last series of experiments was carried out for the HFindMIB algorithm. The results of the algorithm are presented in Table V.

The important indicators for network analysis are the bicliques sizes and their number. These data for each network are presented in Table VI.

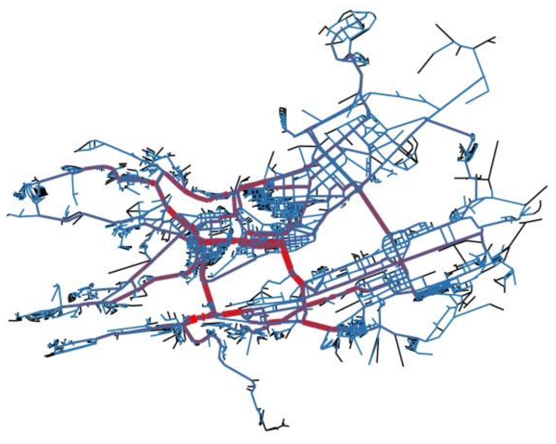

Fig. 4. Heat map of KJA network based on results of TDSP problem solution

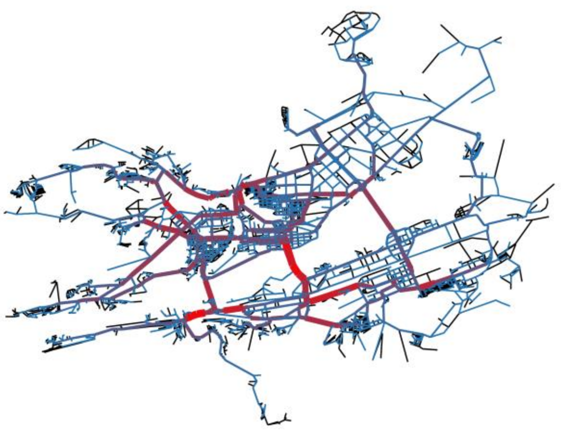

Fig. 5. Heat map of KJA network based on results of RCSP problem solution

TABLE V. RESULTS FOR THE HFINDMIB ALGORITHM

Graph	*n*	*m*	Δ	MBC	Time, sec.
TOF	3840	5227	5	4084	26.527
KJA	4362	6005	5	4823	36.898
NSK	7357	10218	6	7927	103.115
GYD	12458	16417	6	11844	252.244

TABLE VI. NUMBER OF BICLIQUES OF EACH DIMENSION

($\|S_0\|$, $\|S_1\|$)	TOF	KJA	NSK	GYD
(1, 1)	1	3	5	2
(2, 1)	955	1454	1612	2247
(2, 2)	589	535	1204	1154
(3, 1)	1981	2315	3886	7217
(3, 2)	2	–	1	–
(4, 1)	553	514	1215	1215
(5, 1)	3	2	4	9

Using the KJA network as an example, we will present the interpretation of the network elements according to the size of the bicliques. Maximal bicliques with size (2, 1) can be considered as a long road or bypass road. Such bicliques are shown at fig. 6. Different bicliques are marked with different colors.

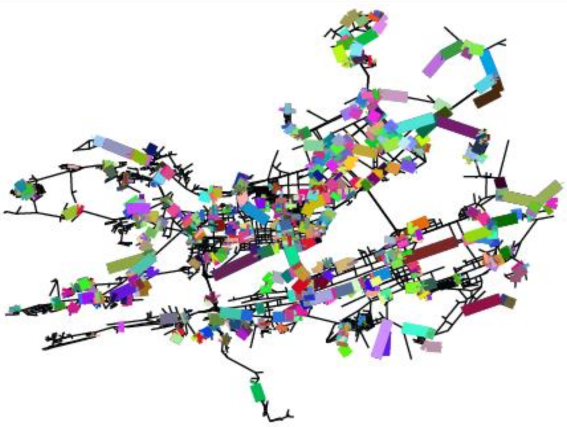

Fig. 6. Map of KJA network with (2, 1) bicliques

Maximal bicliques with size (2, 2) most often correspond to city blocks, which is easy to see in fig. 7. Similar to the previous figure, different bicliques are represented by different colors.

Crossroads and roundabouts are represented with maximal bicliques where the cardinality of one part is equal one and the other part is at least three.

Based on the solutions to the problems indicated in Table 2, an analysis of road networks can be carried out. The results of this analysis can be used to simulate network congestion. Based on information about the most requested edges in the network, one can reorganize and model new bypass roads.

We are present visualization of results of the GYD network without detail analysis. Heat maps for the GYD network are represented at fig. 8 and 9.

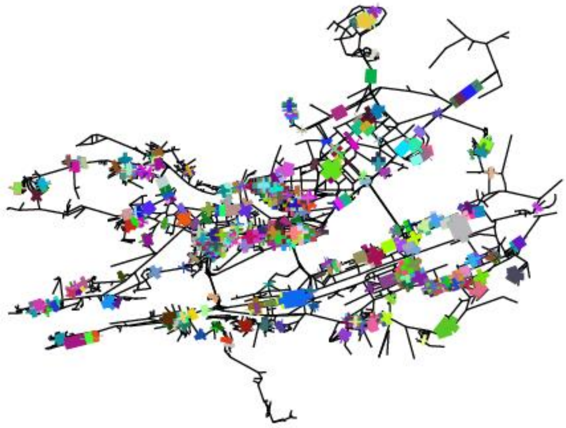

Fig. 7. Map of KJA network with (2, 2) bicliques

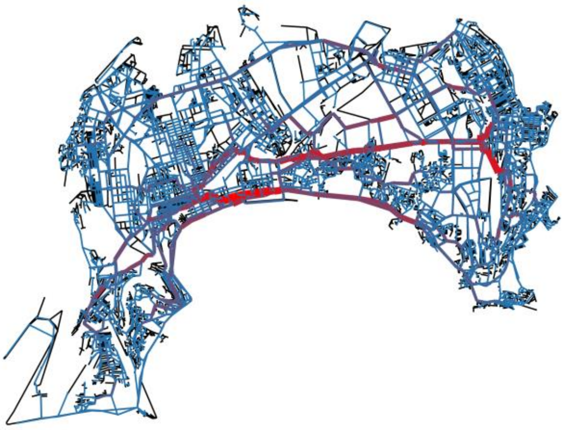

Fig. 8. Heat map of GYD network based on results of TDSP problem solution

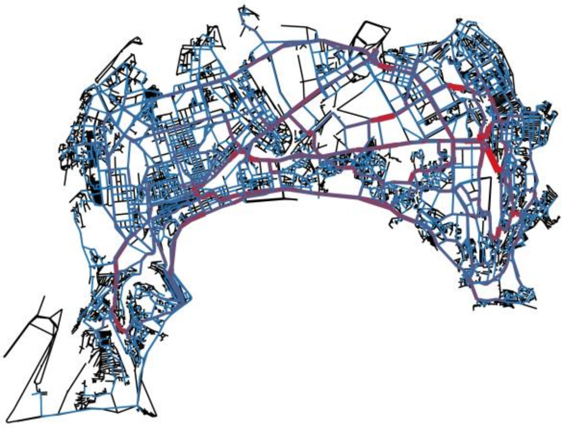

Fig. 9. Heat map of GYD network based on results of RCSP problem solution

Note that percentage of completed requests for the RCSP problem for the GYD network is only 0.539. Thus, the visualization at fig. 9 may not be as informative as at fig. 8. However, even so, there are significant similarities between this heat maps.

Maps of the GYD network with (2, 1) and (2, 2) bicliques are presented at fig. 10 and 11.

Fig. 10. Map of GYD network with (2, 1) bicliques

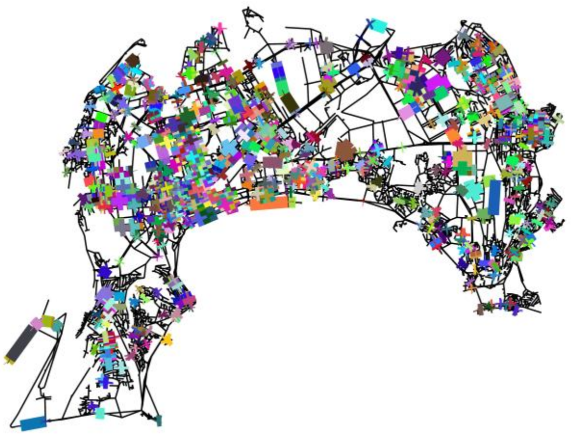

Fig. 11. Map of GYD network with (2, 1) bicliques

IV. CONCLUSION

The paper considers two combinatorial optimization problems: the shortest path problem and the problem of finding all the maximal induced bicliques. Two extensions of the shortest path problem to the cases of a time-dependent metric network with FIFO condition and a resource constrained network are considered. Computational experiments have been carried out for the following algorithms: modified ALT algorithm, RevTree algorithm, HFindMIB algorithm. Based on the results obtained, an analysis of the road network was accomplished.

According to the results obtained, it can be seen that further development of methods for analyzing road networks is promising. These methods can be based on various problems of the shortest path and the problem of finding the maximal induced bicliques.

REFERENCES

[1] V. V. Bykova, and A. A. Soldatenko, “Optimal routing by landmarks in the time-dependent networks,” PDM, vol. 37, pp. 114–123, 2017. (in Russian)

[2] A. A. Soldatenko, “On accuracy of approximation for the resource constrained shortest path problem,” Journal of Siberian Federal University. Mathematics & Physics, vol. 12(5), pp. 621–627, 2019.

[3] A. A. Soldatenko, and D. V. Semenova, “On Problem of Finding all Maximal Induced Bicliques of Hypergraph,” Journal of Siberian Federal University. Mathematics & Physics, vol. 14(5), pp. 638–646, 2021. (in press)

[4] M. Garey, and D. Johnson, Computers and Intractability, Moscow: Mir, 1982. (in Russian)

[5] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 3rd ed., Moscow: «Vil’yams», 2013. (in Russian)

[6] S. Russell, and P. Norvig, Artificial Intelligence: A Modern Approach, 4th ed., Moscow: «Vil’yams», 2021. (in Russian)

[7] M. M. Nejad, L. Mashayekhy, R. B. Chinnam, and A. Phillips, “Hierarchical time-dependent shortest path algorithms for vehicle routing under ITS,” J. IIE Transactions, vol. 48, pp. 158–169, 2016.

[8] D. Delling. and D. Wagner, “Landmark-based Routing in Dynamic Graphs,” In Proc. International Workshop on Experimental Algorithms (WEA), Springer, 2007, 4525 of LNCS, pp. 52–65.

[9] D. Delling, “Time-Dependent SHARC-Routing,” J. Algorithmica, vol. 60, pp. 60–94, 2011.

[10] D. Wagner, and T. Willhalm, “Speed-up techniques for shortest-path computations,” In Proc. STACS, Springer, 2007, 4393 of LNCS, pp. 23–36.

[11] A. Goldberg, H. Kaplan, and R. Werneck, “Better landmarks within reach,” In Proc. International Workshop on Experimental Algorithms (WEA). Springer, 2007, 4525 of LNCS, pp. 38–51.

[12] A. Goldberg, H. Kaplan, and R. Werneck, “Reach for A*: Efficient point-to-point shortest path algorithms,” Technical Report MSR-TR-2005-132, Microsoft Research. Microsoft Corporation, 2005.

[13] H. C. Joksch, “The Shortest Route Problem with Constraints,” Mathematical Analysis and Applecation, vol. 14, pp. 191–197, 1966.

[14] D. Eppstein, “Finding the k shortest paths,” Proceedings 35th Annual Symposium on Foundations of Computer Science, 1994, pp. 154–165.

[15] P. Van Mieghem, F. A. Kuipers, T. Korkmaz, M. Krunz, M. Curado, E. Monteiro, and et al., “Quality of Service Routing,” In Quality of Future Internet Services, 2003, 2856 of LNCS, pp. 80–117.

[16] I. Dumitrescu, and N. Boland, “Improved preprocessing, labeling and scaling algorithms for the weight-constrained shortest path problem,” Networks, vol. 42, pp. 135–153, 2003.

[17] R. Hassin, “Approximation Schemes for the Restricted Shortest Path Problem,” Mathematics Oper. Res., vol. 17, pp. 36–42, 1992.

[18] X. Zhu, and W. E. Wilhelm, “A three-stage approach for the resource-constrained shortest path as a sub-problem in column generation,” Computers & Operations Research, vol. 39(2), pp. 164–178, 2012.

[19] K. Mehlhorn, and M. Ziegelmann, “Resource constrained shortest paths,” In Algorithms ESA, 2000, 1879 of LNCS, pp. 326–337.

[20] M. Horvath, and T. Kis, “Solving resource constrained shortest path problems with LP-based methods,” Computers, vol. 73, pp. 150–164, 2016.

[21] M. Jepsen, B. Petersen, S. Spoorendonk, and D. Pisinger, “A branch-and-cut algorithm for the capacitated profitable tour problem,” Discrete Optimization, vol. 14, pp. 78–96, 2014.

[22] Y. Bejerano, Y. Breitbart, A. Orda, R. Rastogi, and A. Sprintson, “Algorithms for computing QoS paths with restoration,” IEEE/ACM Transactions on Networking, vol. 13(3), pp. 648–661, 2005.

[23] G. Xue, W. Zhang, J. Tang, and K. Thulasiraman, “Polynomial Time Approximation Algorithms for Multi-Constrained QoS Routing,” IEEE/ACM Transactions on Networking, vol. 16(3), pp. 656–669, 2008.

[24] L. Di Puglia Pugliese, and F. Guerriero, “A survey of resource constrained shortest path-problems: exact solution approaches,” Networks, pp. 183–200, 2013.

[25] B. Lyu, L. Qin, X. Lin, Y. Zhang, Z. Qian, and J. Zhou, “Maximum biclique search at billion scale,” Proc. VLDB Endow, 2020, pp. 1359–1372.

[26] G. Alexe, S. Alexe, Y. Crama, S. Foldes, P. L. Hammer, and B. Simeone, “Consensus algorithms for the generation of all maximal bicliques,” Discrete Applied Mathematics, vol. 145, pp. 11–21, 2004.

[27] E. Prisner, “Bicliques in graphs I: bounds on their number,” Combinatorica, vol. 20, pp. 109–117, 2000.

[28] D. Hermelin, and G. Manoussakis, “Efficient enumeration of maximal induced bicliques,” Discrete Applied Mathematics, vol. 303, pp. 253–261, 2021.

[29] J. Li, G. Liu, H. Li, and L. Wong, “Maximal Biclique Subgraphs and Closed Pattern Pairs of the Adjacency Matrix: A One-to-One

Correspondence and Mining Algorithms," IEEE Transactions on Knowledge and Data Engineering, vol. 19(12), pp. 1625–1673, 2007.

[30] P. Damaschke, "Enumerating maximal bicliques in bipartite graphs with favorable degree sequences," Inf. Process. Lett., vol. 114(6), pp. 317–321, 2014.

[31] DIMACS Implementation Challenge (2006). Shortest Paths: road graphs database. [Online]. Avaible: http://www.dis.uniroma1.it/challenge9/

[32] G. Boeing, "OSMnx: New Methods for Acquiring, Constructing, Analyzing, and Visualizing Complex Street Networks," Computers, Environment and Urban Systems, vol. 65, pp. 126–139, 2017.

[33] V. V. Bykova, and A. A. Soldatenko, "Adaptive landmark selection in shortest-paths problem for a big graph," J. Software & Systems, vol. 1(113), pp. 60–67, 2016. (in Russian)